# Theory of nuclear reactions

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# theory opportunities with FRIB

**DOE Nuclear Physics Mission is to understand the fundamental forces and particles of nature as manifested in nuclear matter, and provide the necessary expertise and tools from nuclear science to meet national needs**

**DOE Nuclear Physics Mission is accomplished by supporting scientists who answer overarching questions in major scientific thrusts of basic nuclear physics research**

| Science Drivers (Thrusts) from NRC RISAC | | | |
|---|---|---|---|
| Nuclear Structure | Nuclear Astrophysics | Tests of Fundamental Symmetries | Applications of Isotopes |
| Overarching Questions from NSAC 2007 LRP | | | |
| ★ What is the nature of the nuclear force that binds protons and neutrons into stable nuclei and rare isotopes?<br>★ What is the origin of simple patterns in complex nuclei? | What is the nature of neutron stars and dense nuclear matter? ★<br>What is the origin of the elements in the cosmos? ★<br>What are the nuclear reactions that drive stars and stellar explosions? ★ | Why is there now more matter than antimatter in the universe? | What are new applications of isotopes to meet the needs of society? |

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**Overarching questions are answered by rare isotope research**

17 Benchmarks from NSAC RIB TF measure capability to perform rare isotope research

| Nuclear Structure | Nuclear Astrophysics | Tests of Fundamental Symmetries | Applications of Isotopes |
|---|---|---|---|
| Shell structure<br>2. Superheavies<br>3. Skins<br>Pairing<br>5. Symmetries<br>Limits of stability<br>Weakly bound nuclei<br>15. Mass surface | 6. Equation of State (EOS)<br>r-Process<br>8. $^{15}O(\alpha,\gamma)$<br>9. $^{59}Fe$ supernovae<br>15. Mass surface<br>16. rp-Process<br>17. Weak interactions | 12. Atomic electric dipole moment | 10. Medical<br>11. Stewardship |

FRIB-CDR, 2010

# Theory of nuclear reactions: outline

1. Some basics:
   - Classification, motivation, definitions
2. Single channel scattering:
   - S-matrix, phase shift, T-matrix, scattering amplitude, resonances
   - including Coulomb
3. Optical potential and absorption
4. Multi-channel equation
   - reaction cross section
   - detailed balance
5. Integral forms
   - Lipmann-Schwinger Equation,
   - two potential formula,
   - Distorted wave Born approximation
6. Three-body methods:
   - Faddeev, Continuum Discretized Coupled Channel, Adiabatic Wave Approximation
7. Perspectives

# classification of reactions

Direct reactions

transfer momentum is small compared to initial momentum
typically peripheral
short timescale ($10^{-22}$ s)
E>10 MeV
mostly one step
final states keep memory of initial states

Resonance reactions

reactions that go through a resonance (peak in the cross section)
intermediate step in the reaction
longer timescale (depends of lifetime of resonance)

Compound reactions

longer timescale
many steps in the reaction
all nucleons share the beam energy
loss of memory from the initial state
low energy reactions

# Elastic scattering

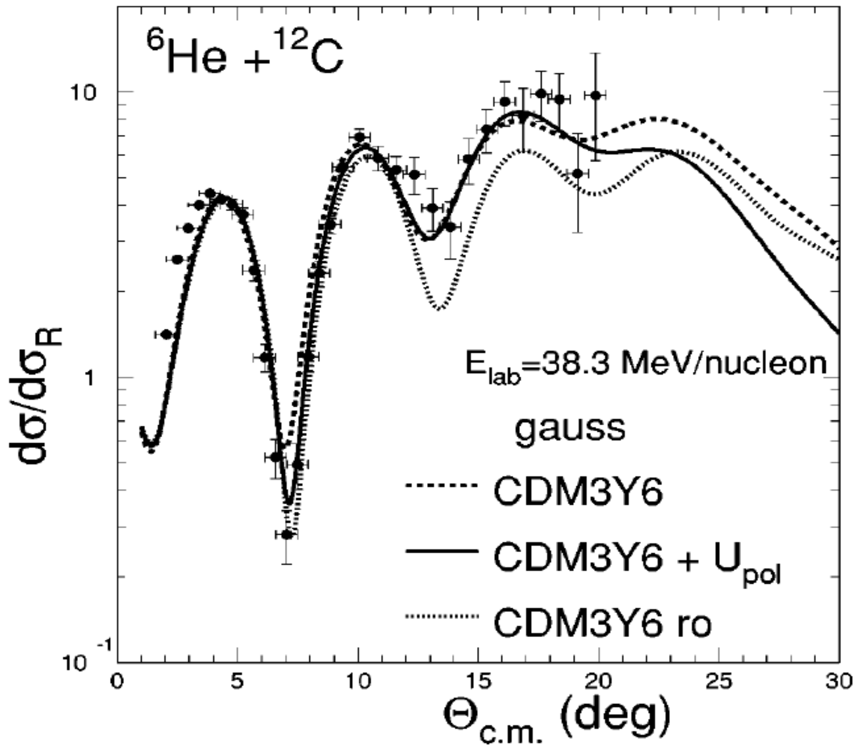
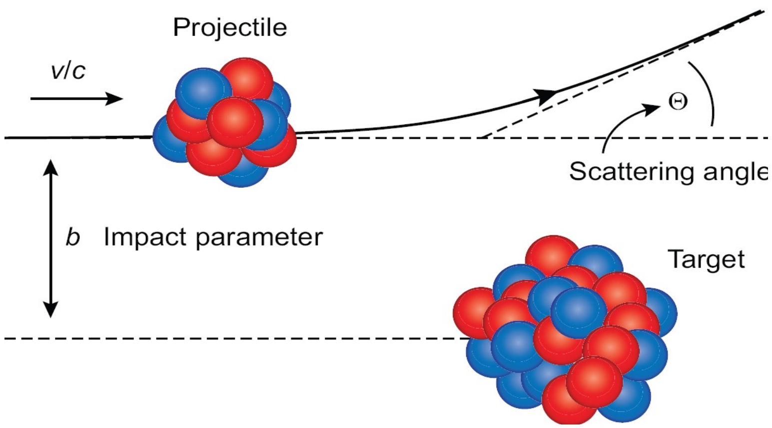



FIG. 10. Elastic scattering for $^6$He+ $^{12}$C at 38.3 MeV/nucleon in comparison with the OM results given by the real folded potential (obtained with the CDM3Y6 interaction and the Gaussian *ga* density for $^6$He). The dashed curve is obtained with the unrenormalized folded potential only. The solid curve is obtained by adding a complex surface polarization potential to the real folded potential. Its parameters, and those of the imaginary part, are explained in the text. The dotted line is obtained by folding the CDM3Y6 interaction with the compact Gaussian density *ro*.

[Lapoux et al, PRC 66 (02) 034608]

***traditionally used to extract optical potentials, rms radii, density distributions.***

# Inelastic excitation



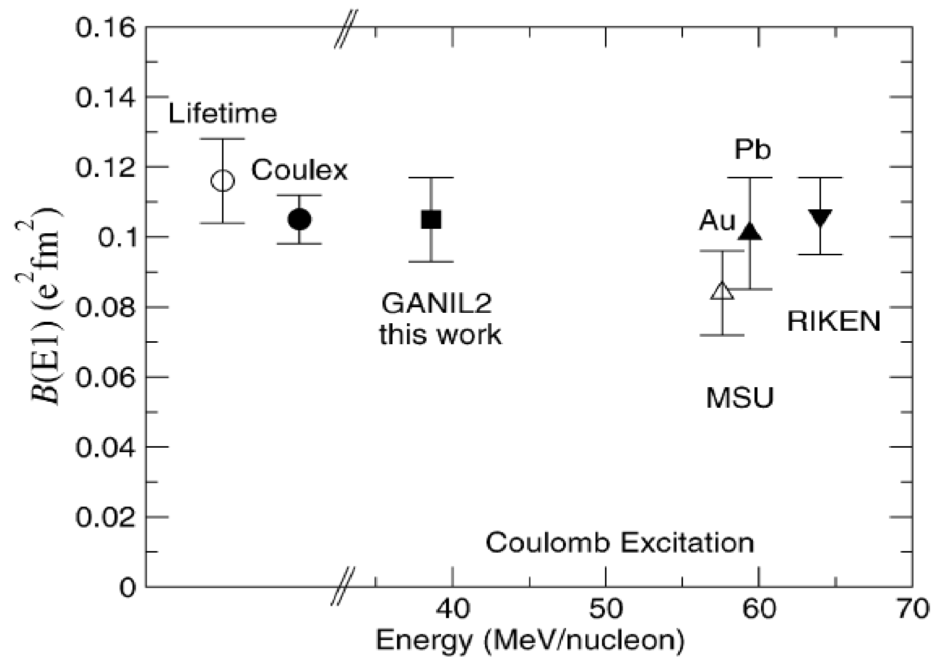
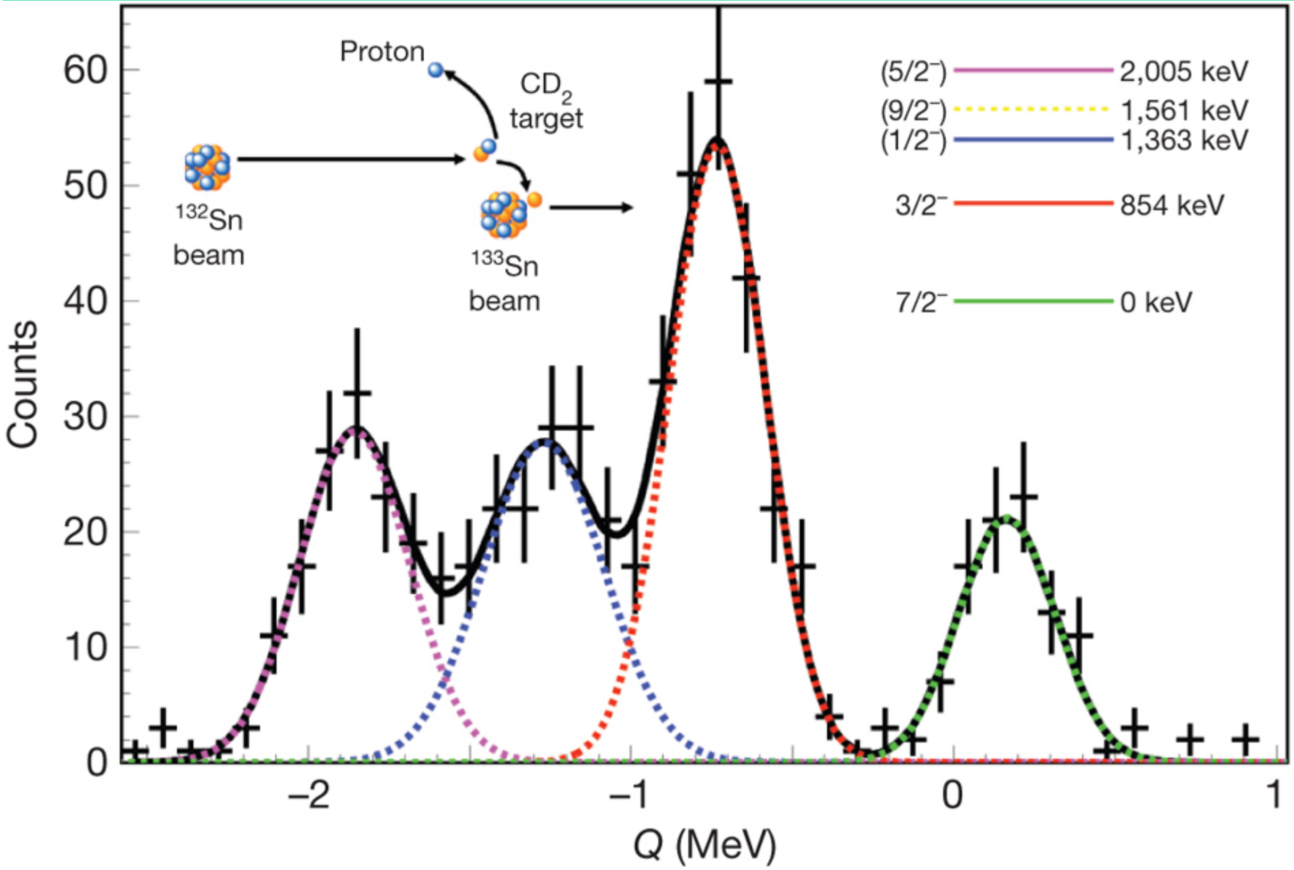




Fig. 2. Comparison of $B$(E1) values obtained from lifetime and Coulomb excitation measurements. The weighted average of lifetime measurements [3] (open circle) is plotted on the left along with the weighted average (solid circle) of three Coulomb excitation measurements (solid symbols). The individual Coulomb excitation measurements, GANIL (this work, square), MSU (up triangle) [6], RIKEN (down triangle) [7], and a previous GANIL experiment (diamond) [4], are plotted versus the beam energy.

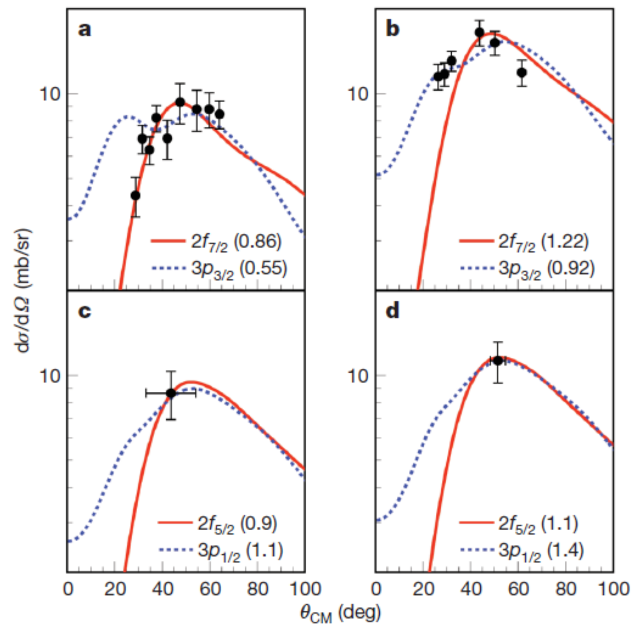
***traditionally used to extract electromagnetic transitions or nuclear deformations***

[Summers et al, PLB 650 (2007) 124]

# Transfer reactions




**d($^{132}$Sn,$^{133}$Sn)p@5 MeV/u**




***traditionally used to study shell structure: extract spin,parity and spectroscopic factors***



[K. Jones et al, Nature 465 (2010) 454]

# Two-nucleon transfer

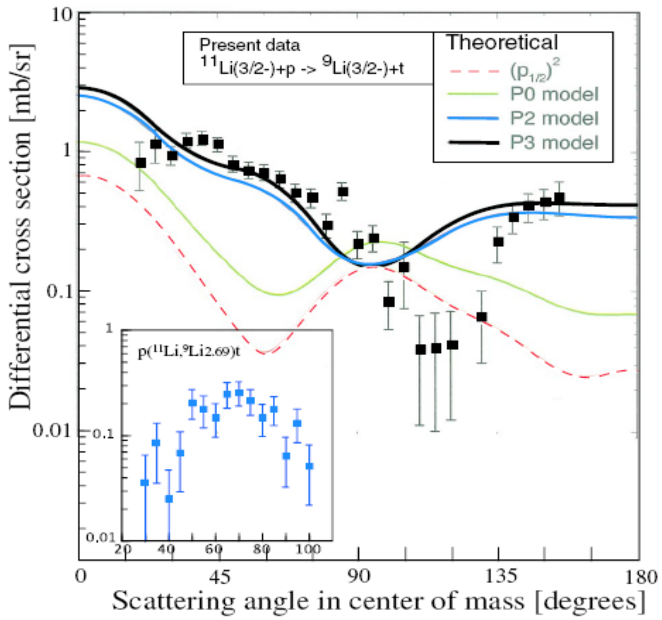



FIG. 3 (color online). Differential cross sections of the $(p, t)$ reaction to the ground state of $^{9}$Li and to the first excited state (insert). Theoretical predictions using four different wave functions were shown by curves. See the text for the difference of the wave functions.

$^{11}$Li(p,t)$^{9}$Li@ 3 A MeV

measured both ground state and excited state $^{9}$Li

[Tanihata et al, PRL 100, 192502 (2008)]

***traditionally used to study two nucleon correlations and pairing***

# Knockout reactions

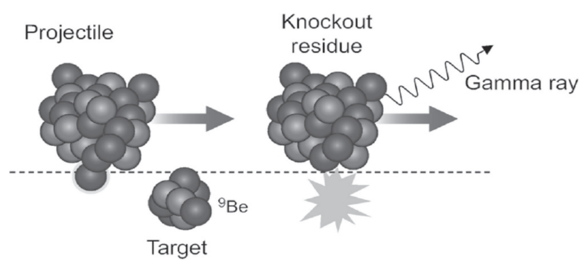
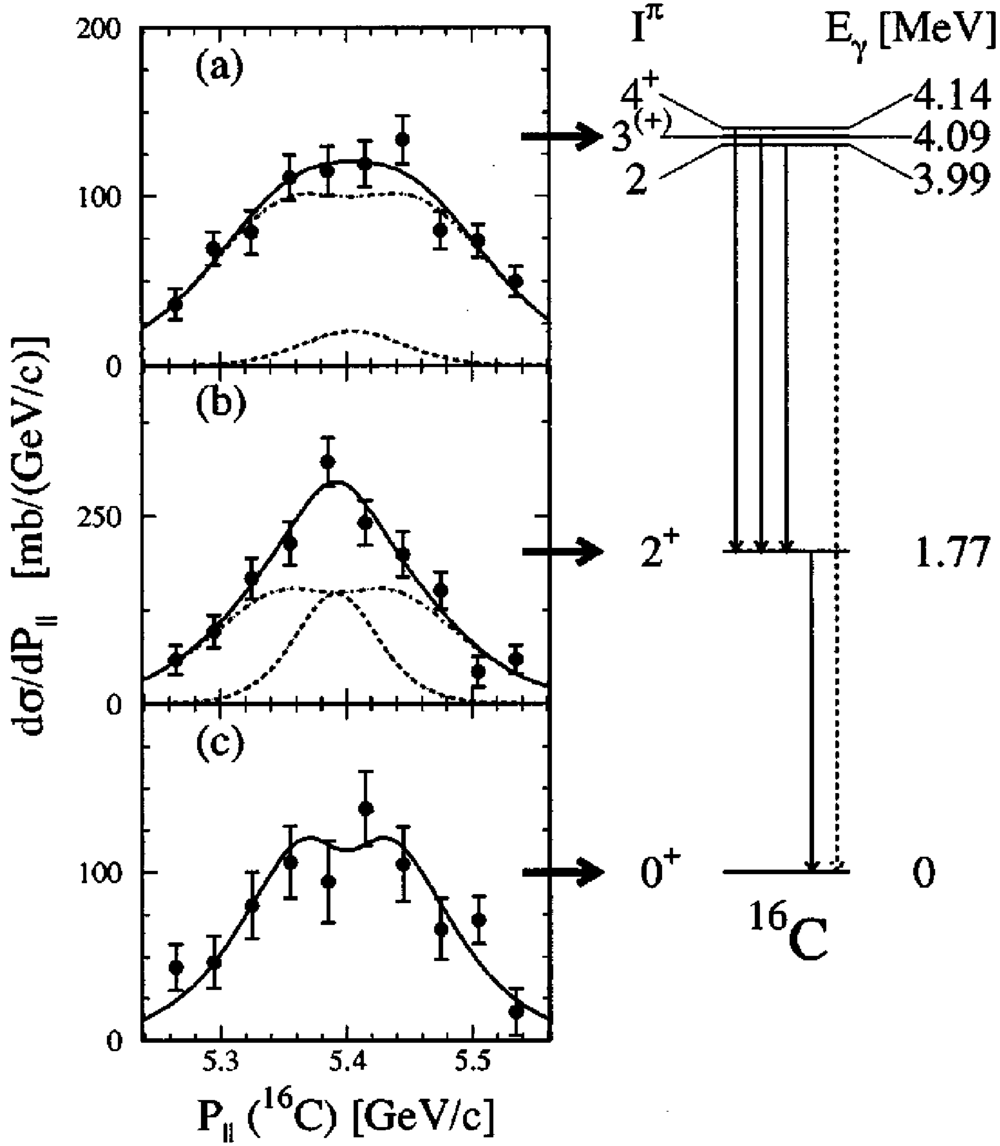



Fig. 14.9. Schematic of a nuclear knockout reaction. Reprinted from [3] with permission.

**Includes elastic and inelastic breakup as well as transfer**

***traditionally used to study shell structure***




**Maddalena et al.,** PRC63(01)024613

# Breakup reactions

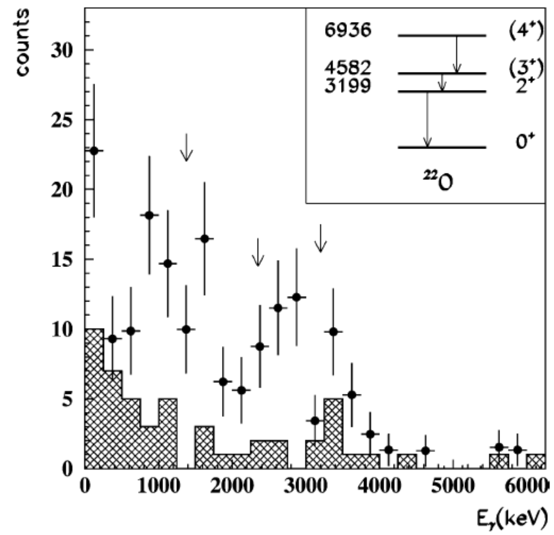
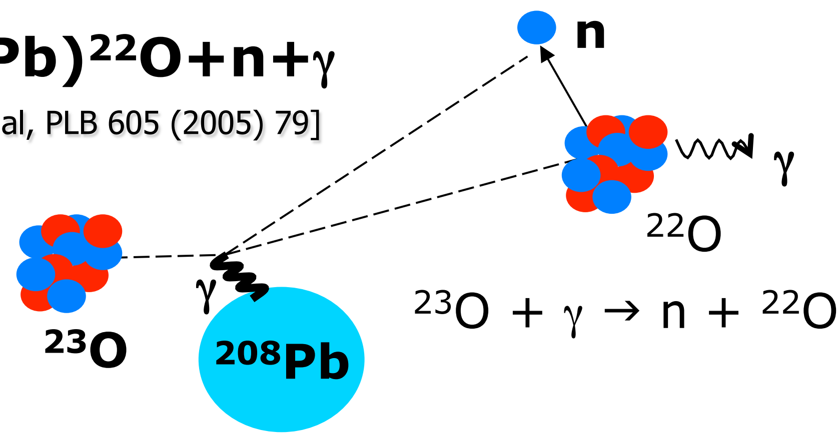



Fig. 1. Doppler corrected $\gamma$-ray spectra measured in coincidence with an $^{22}O$ fragment and one neutron for Pb (symbols) and C (shaded area) targets. Arrows indicate the strongest $\gamma$ transitions as expected from the $^{22}O$ level scheme of Ref. [10] (partial level scheme shown as inset; level energies are in keV).

**$^{23}O(Pb,Pb)^{22}O+n+\gamma$**

[Nociforo et al, PLB 605 (2005) 79]




***traditionally used to study halos, states in the continuum, and transition strengths to bound states***

# Charge exchange reactions

PHYSICAL REVIEW C **74**, 034333 (2006)

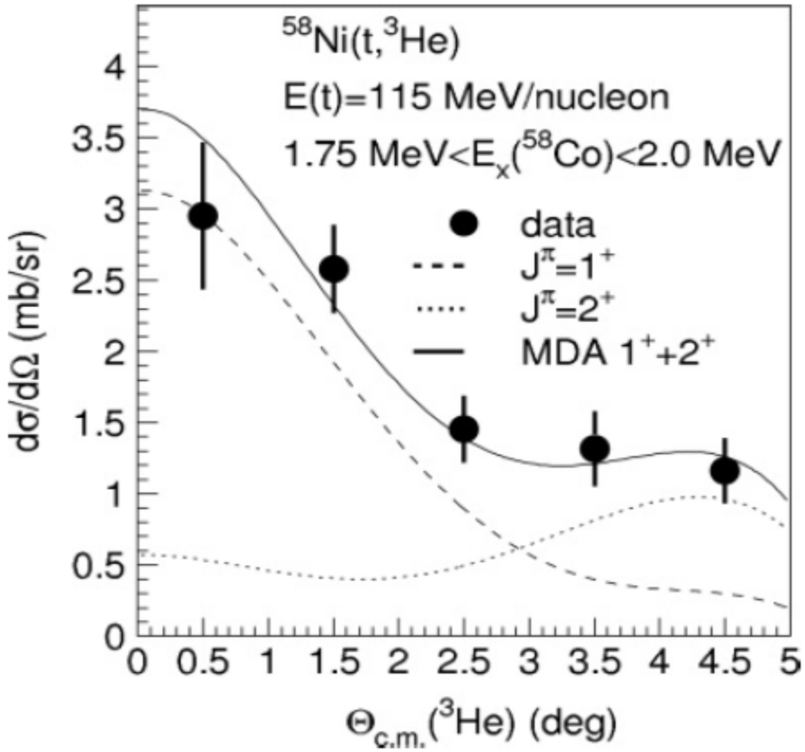



FIG. 5. The differential cross sections for the energy bin $1.75 < E_x(^{58}\mathrm{Co}) < 2.0$ MeV and the result of the MDA (solid line) using a linear combination of $1^+$ (dashed line) and $2^+$ (dotted line) components. The error bars in the data are of statistical nature only.

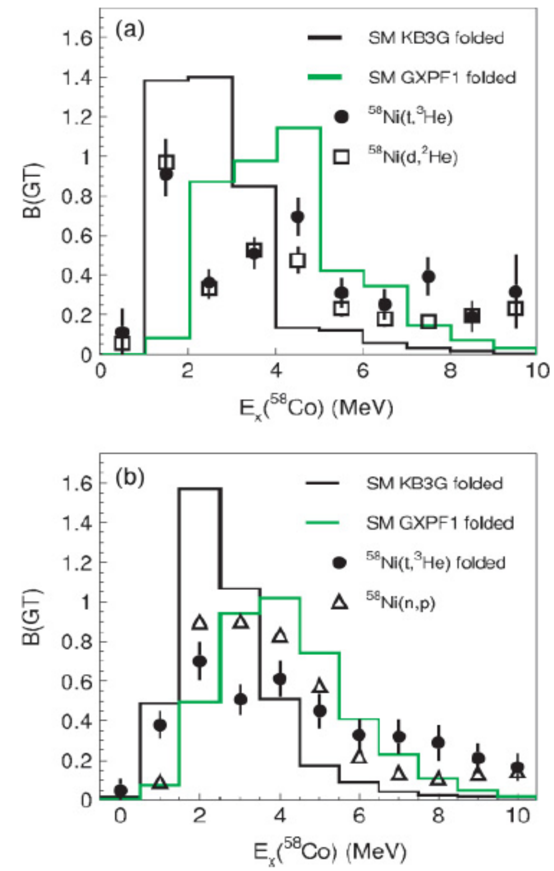



FIG. 7. (Color online) (a) Comparison of the results of the $^{58}$Ni$(d,{}^2\mathrm{He})$ and $^{58}$Ni$(t,{}^3\mathrm{He})$ experiments and the theoretical predictions. A binning of 1 MeV was applied and the theory was folded with the experimental resolution of the $(t,{}^3\mathrm{He})$ experiment (250 keV) before binning. (b) Comparison of the results of the $^{58}$Ni$(n,p)$ and $^{58}$Ni$(t,{}^3\mathrm{He})$ experiments and the theoretical predictions. A binning of 1 MeV was applied. Note the 0.5-MeV shift relative to a). The $(t,{}^3\mathrm{He})$ data set and theory were folded with the experimental resolution of the $(n,p)$ experiment (1.3 MeV) before binning.

***traditionally used to study Gamow Teller transitions***

# Fusion reactions

## Fusion of Stable vs Unstable Nuclei

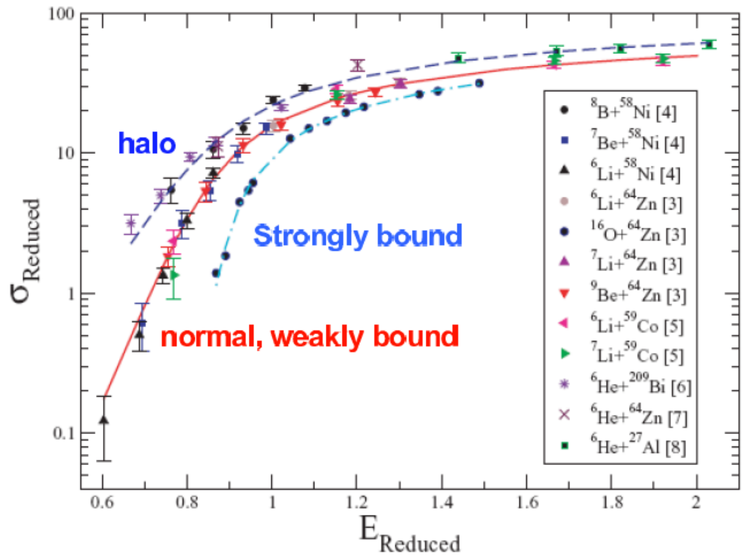
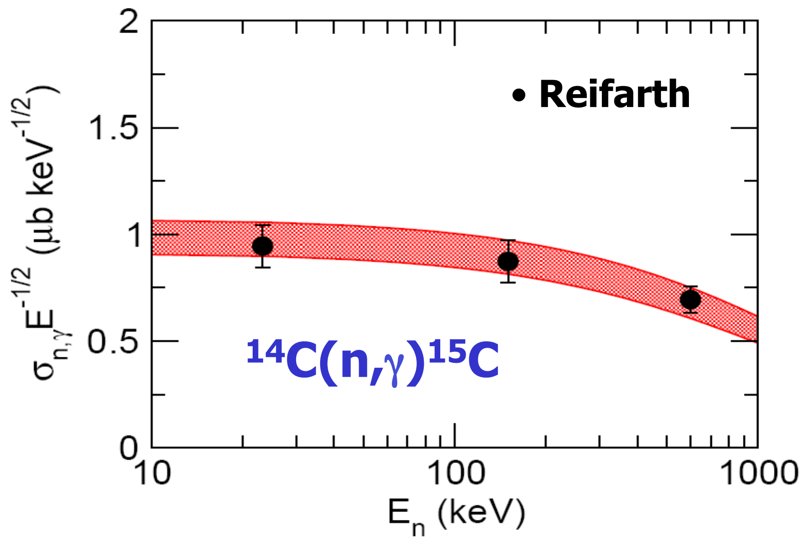



Fig. 8. Reduced cross sections for the fusion of halo, normal/weakly bound, and strongly bound nuclei. (Courtesy of Kolata).

After geometric effects are scaled out, fusion enhanced for halo nuclei!

*Superheavies*
*Halos*
*Applications: energy*

# Capture reactions

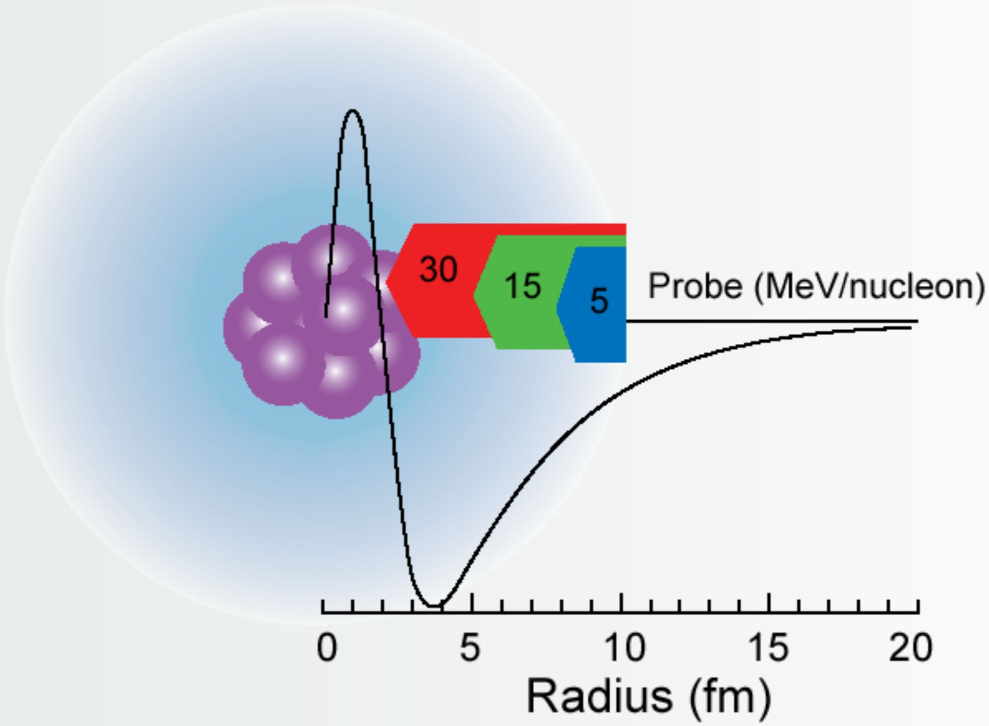



*Typically measured with astrophysical motivation*



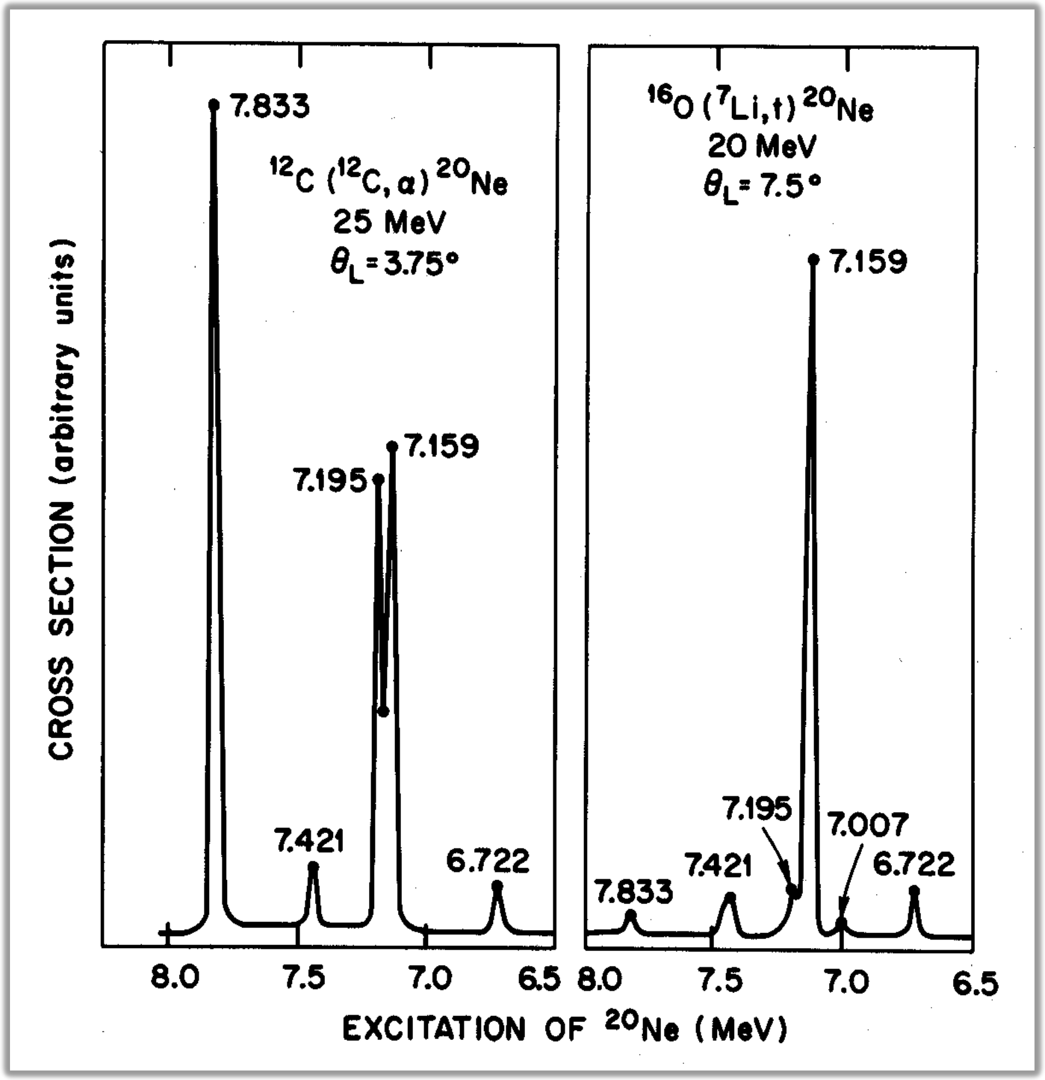
Summers and Nunes, PRC78(2009)069908

# direct reactions and tomography




The overlap function for $^{19}C \rightarrow n+^{18}C$ in arbitrary units. The radial sensitivity of the $^{18}C(d,p)^{19}C$ cross section is represented by the colored bars for different beam energies.

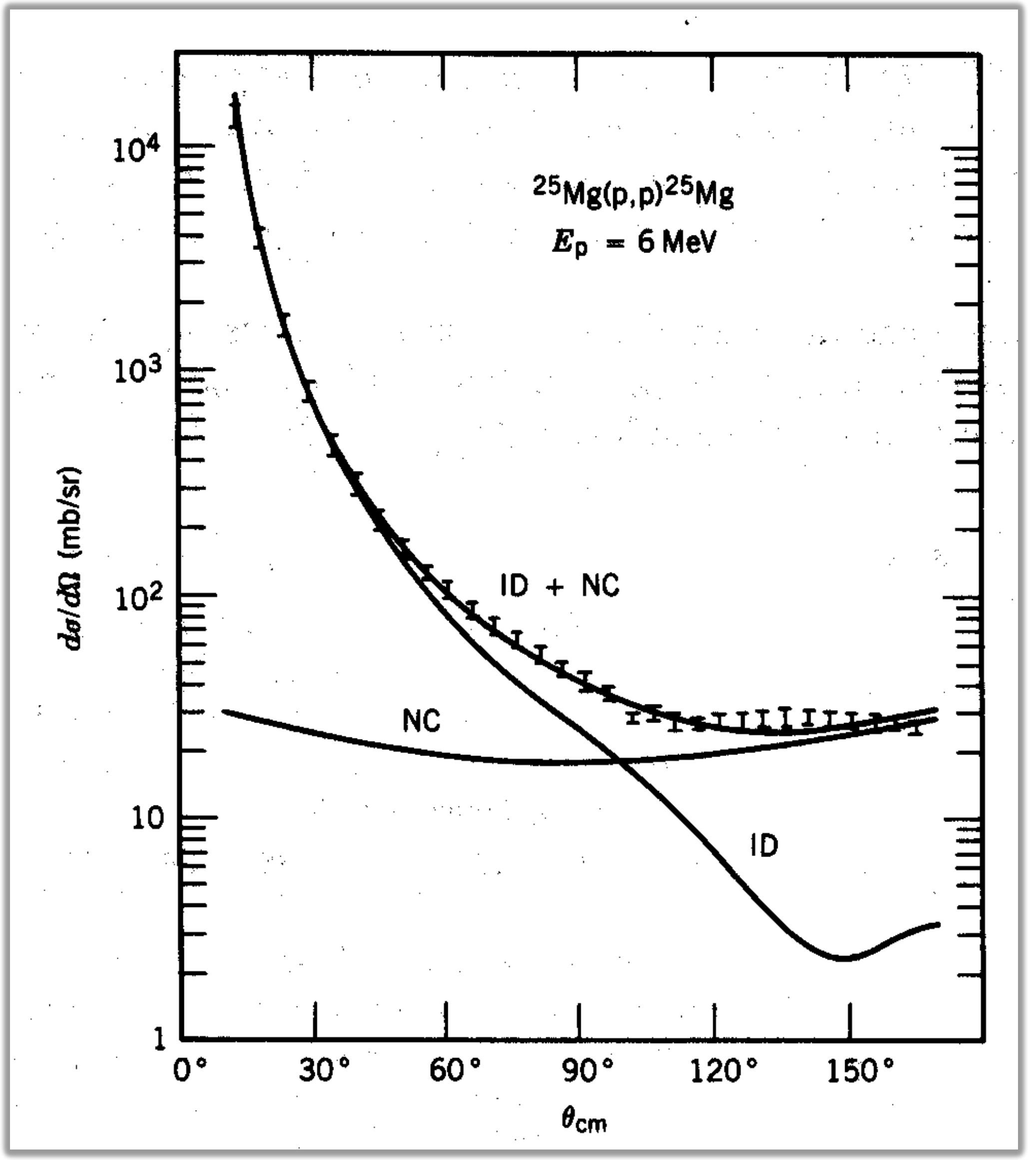
# selectivity of the reaction to resonances



# angular distribution: compound vs direct

Direct reactions (ID):
Forward peaked (large b)

Compound reactions (NC):
Distribution is generally isotropic (except for heavy ion collision where L large)



# equations of motion

**laboratory**

$$\left[-\frac{\hbar^2}{2m_A}\nabla^2_{\mathbf{r}_A} - \frac{\hbar^2}{2m_B}\nabla^2_{\mathbf{r}_B} + V(\mathbf{r}_A - \mathbf{r}_B) - E_{\text{tot}}\right]\Psi(\mathbf{r}_A, \mathbf{r}_B) = 0.$$

**Center of mass**

$$\left[-\frac{\hbar^2}{2m_{AB}}\nabla^2_{\mathbf{S}} - \frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{R}} + V(\mathbf{R}) - E_{\text{tot}}\right]\Psi(\mathbf{S}, \mathbf{R}) = 0.$$

$$\Psi(\mathbf{S}, \mathbf{R}) = \Phi(\mathbf{S})\psi(\mathbf{R})$$

$$\Phi(\mathbf{S}) = A\exp(\mathrm{i}\mathbf{K}\cdot\mathbf{S})$$

$$\boxed{\begin{array}{l} -\dfrac{\hbar^2}{2m_{AB}}\nabla^2_{\mathbf{S}}\Phi(\mathbf{S}) = (E_{\text{tot}} - E)\,\Phi(\mathbf{S}) \\ \text{and} \quad \left[-\dfrac{\hbar^2}{2\mu}\nabla^2_{\mathbf{R}} + V(\mathbf{R})\right]\psi(\mathbf{R}) = E\,\psi(\mathbf{R}) \end{array}}$$

# cross section

The number of particle entering a detector depends on:

- solid angular size of detector
- number of scattering centers in the target
- flux of the incident beam
- the cross sectional area for the reaction to occur

$$\boxed{\frac{dN}{dt} = j_i \; n \; \Delta\Omega \; \sigma}$$

$$\mathbf{j} = \mathbf{v}|\psi|^2$$

# cross section

<u>Definition of cross section:</u>

the area within which a projectile and a target will interact and give rise to a specific product.

<u>Units</u> 1b (barn) = 10 fm x 10 fm

If we consider just one scattering center $n = 1$, and measure the scattered *angular flux* in the final state as $\hat{j}_f(\theta, \phi)$ particles/second/steradian, then

$$\boxed{\sigma(\theta, \phi) = \frac{\hat{j}_f(\theta, \phi)}{j_i}}$$

# cross section in c.m. and lab

Total cross section:

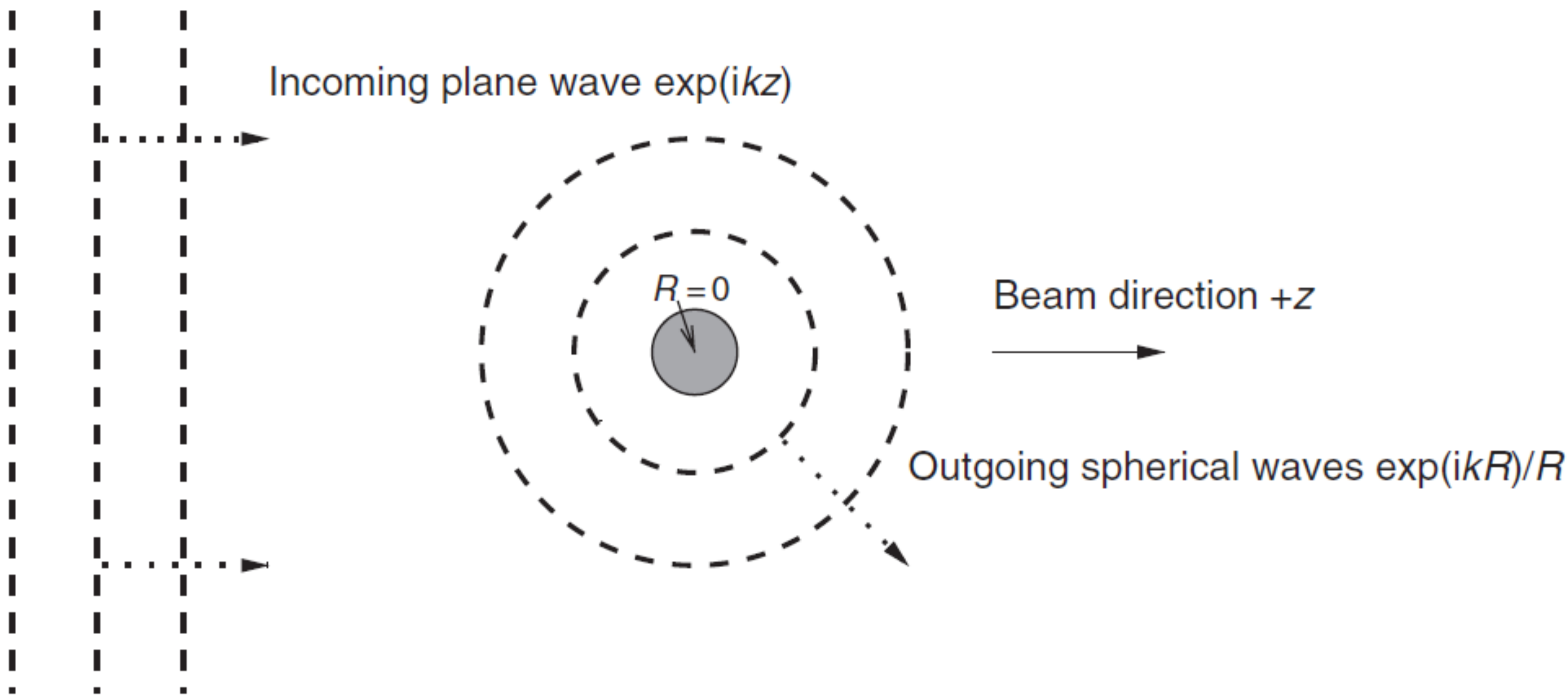
the same in center of mass and laboratory

Angular distribution of the cross section:

$$\sigma(\theta,\phi)\,\mathrm{d}\phi\ \sin\theta\,\mathrm{d}\theta = \sigma_{\mathrm{lab}}(\theta_{\mathrm{lab}},\phi_{\mathrm{lab}})\,\mathrm{d}\phi_{\mathrm{lab}}\ \sin\theta_{\mathrm{lab}}\mathrm{d}\theta_{\mathrm{lab}}$$

# Scattering theory: single channel

# picture for scattering



# Scattering theory: single channel

Incoming beam

$$\psi^{\text{beam}} = A\exp(\mathrm{i}\mathbf{k}_i\cdot\mathbf{R})$$

$$\psi^{\text{beam}} = A\mathrm{e}^{\mathrm{i}k_i z}$$

Incoming flux

$$k = \sqrt{2\mu E/\hbar^2}$$

$$\mathbf{v} = \mathbf{p}/\mu = \hbar\mathbf{k}/\mu$$

$$j_i = v_i|A|^2$$

Scattered wave

$$\psi^{\text{scat}} = Af(\theta,\phi)\frac{\mathrm{e}^{\mathrm{i}k_f R}}{R}$$

Outgoing flux

$$j_f = v_f|A|^2|f(\theta,\phi)|^2/R^2$$

Asymptotic wave

$$\psi^{\text{asym}} = \psi^{\text{beam}} + \psi^{\text{scat}} = A\left[\mathrm{e}^{\mathrm{i}k_i z} + f(\theta,\phi)\frac{\mathrm{e}^{\mathrm{i}k_f R}}{R}\right]$$

Scattering amplitude

# Scattering theory: single channel

$$k = \sqrt{2\mu E/\hbar^2}$$
$$\mathbf{v} = \mathbf{p}/\mu = \hbar\mathbf{k}/\mu$$

Scattered angular flux and incoming flux

$$\hat{j}_f = R^2 j_f = v_f |A|^2 |f(\theta,\phi)|^2$$
$$j_i = v_i |A|^2$$

Cross section

$$\sigma(\theta,\phi) = \frac{v_f}{v_i}|f(\theta,\phi)|^2$$

Renormalized scattering amplitude

$$\tilde{f}(\theta,\phi) = \sqrt{\frac{v_f}{v_i}} f(\theta,\phi)$$

$$\sigma(\theta,\phi) = |\tilde{f}(\theta,\phi)|^2$$

# Scattering equation

- short range potentials V(**R**)=0, $R>R_n$
  no Coulomb for now
- positive energy time-independent Schrodinger eq to obtain $f(\theta,\phi)$
  numerical solutions matched to asymptotic form
- spherical potentials V(**R**)=V(R)
  angular momentum and energy commute
  initial beam is cylindrically symm (m=0) implies scattered wave is too: $f(\theta,\phi)= f(\theta)$

$$\hat{T} = -\frac{\hbar^2}{2\mu}\nabla_R^2$$
$$= \frac{\hbar^2}{2\mu}\left[-\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial}{\partial R}\right)+\frac{\hat{L}^2}{R^2}\right]$$

$$\boxed{[\hat{T}+V-E]\psi(R,\theta)=0}$$

# Partial wave expansion

- Legendre polynomials form a complete set $\sum_L b_L(R) P_L(\cos\theta)$
- they are eigenstates of $\hat{L}^2$ and $\hat{L}_z$
- orthogonality relation: $\int_0^\pi P_L(\cos\theta) P_{L'}(\cos\theta) \sin\theta \mathrm{d}\theta = \frac{2}{2L+1}\delta_{LL'}$
- particular form for expansion

$$\nabla_R^2 P_L(\cos\theta) \frac{\chi_L(R)}{R} = \frac{1}{R}\left(\frac{\mathrm{d}^2}{\mathrm{d}R^2} - \frac{L(L+1)}{R^2}\right) \chi_L(R) P_L(\cos\theta)$$

- partial wave expansion:

$$\boxed{\psi(R,\theta) = \sum_{L=0}^{\infty} (2L+1) \mathrm{i}^L P_L(\cos\theta) \frac{1}{kR} \chi_L(R)}$$

- partial wave equation:

$$\boxed{\left[-\frac{\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2}{\mathrm{d}R^2} - \frac{L(L+1)}{R^2}\right) + V(R) - E\right] \chi_L(R) = 0}$$

# free solution and coulomb functions

- when V(**R**)=0, for all R

$$\rho \equiv kR$$

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{L(L+1)}{\rho^2} + 1\right]\chi_L^{\mathrm{ext}}(\rho/k) = 0$$

$$\eta = 0$$

- Coulomb wave equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1\right]X_L(\eta,\rho) = 0$$

- two linearly independent solutions:
  regular and irregular Coulomb functions

$$F_L(\eta,\rho) \quad G_L(\eta,\rho)$$

- two linearly independent solutions:
  outgoing and incoming Hanckel functions

$$H_L^{\pm} = G_L \pm \mathrm{i}F_L$$

# Partial wave expansion for plane wave

$$\psi(R,\theta) = \sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR}\chi_L(R)$$

$$\mathrm{e}^{\mathrm{i}kz} = \sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR}F_L(0,kR)$$

$$\mathrm{e}^{\mathrm{i}kz} = \sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)\frac{1}{kR}\frac{\mathrm{i}}{2}[H_L^-(0,kR) - H_L^+(0,kR)]$$

incoming outgoing

- at large distances the radial wavefunction should behave as

$$\chi_L^{\mathrm{ext}}(R) = A_L\left[H_L^-(0,kR) - \mathbf{S}_L H_L^+(0,kR)\right]$$

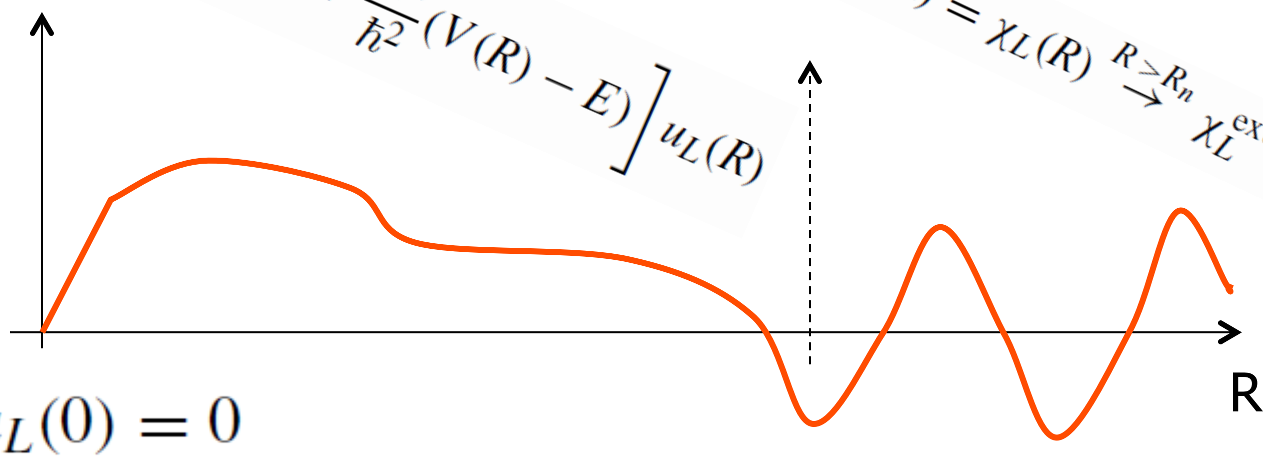
partial wave S-matrix element

# Matching to asymptotics

- numerical solution is proportional to true solution $\chi_L(R) = Bu_L(R)$

$$u_L''(R) = \left[\frac{L(L+1)}{R^2} + \frac{2\mu}{\hbar^2}(V(R) - E)\right]u_L(R)$$

$$Bu_L(R) = \chi_L(R) \xrightarrow{R>R_n} \chi_L^{\rm ext}(R) = A_L\left[H_L^-(0,kR) - \mathbf{S}_L H_L^+(0,kR)\right]$$



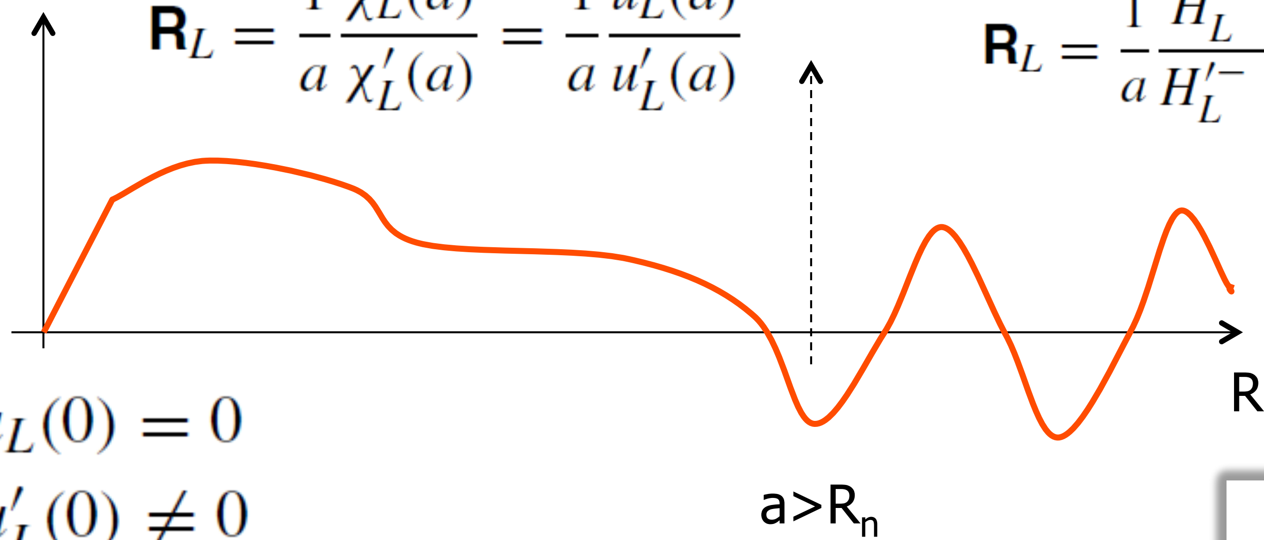

$$u_L(0) = 0$$
$$u'_L(0) \neq 0$$

# Inverse logarithmic derivative

- The matching can be done with the inverse log derivative $R_L$
- any potential will produce $R_L$ which relates to $S_L$

$$\mathbf{R}_L = \frac{1}{a}\frac{\chi_L(a)}{\chi'_L(a)} = \frac{1}{a}\frac{u_L(a)}{u'_L(a)}$$

$$\mathbf{R}_L = \frac{1}{a}\frac{H_L^- - \mathbf{S}_L H_L^+}{H_L'^- - \mathbf{S}_L H_L'^+}$$




$$u_L(0) = 0$$
$$u'_L(0) \neq 0$$

$$\boxed{\mathbf{S}_L = \frac{H_L^- - a\mathbf{R}_L H_L'^-}{H_L^+ - a\mathbf{R}_L H_L'^+}}$$

# S-matrix and scattering amplitude

- to obtain the scattering amplitude need to sum the partial waves

$$\psi(R,\theta) \overset{R>R_n}{\rightarrow} \frac{1}{kR}\sum_{L=0}^{\infty}(2L+1)\mathrm{i}^L P_L(\cos\theta)A_L[H_L^-(0,kR)-\mathbf{S}_L H_L^+(0,kR)]$$

$$\psi^{\mathrm{asym}}(R,\theta) = \mathrm{e}^{\mathrm{i}kz} + f(\theta)\frac{\mathrm{e}^{\mathrm{i}kR}}{R}$$

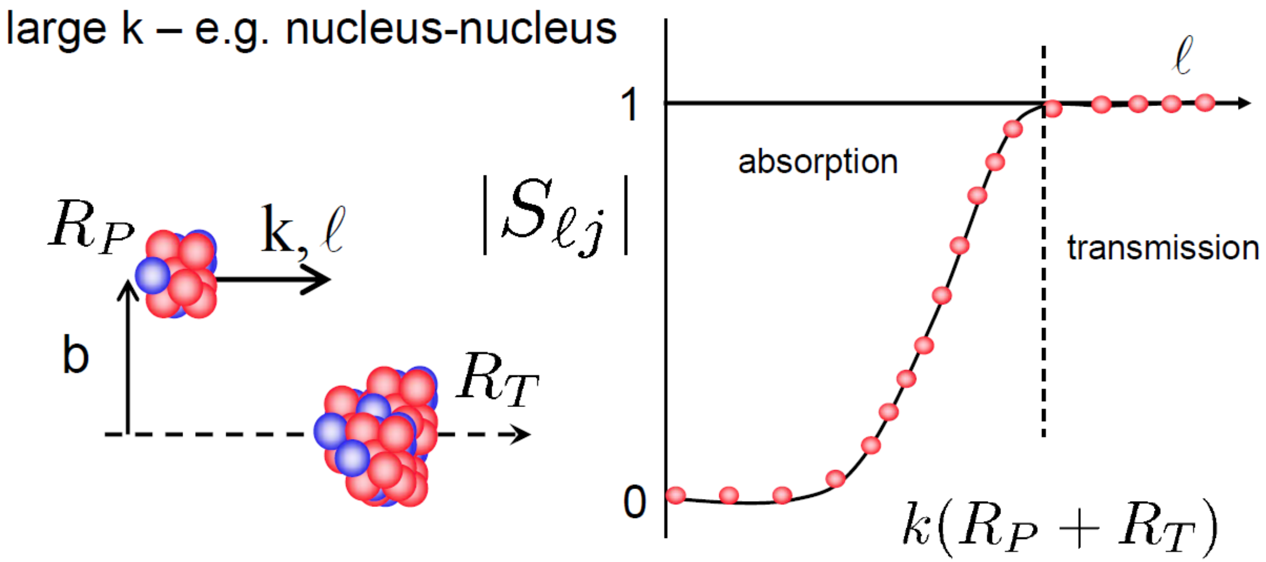
- homework!

$$\boxed{f(\theta) = \frac{1}{2\mathrm{i}k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)(\mathbf{S}_L - 1)}$$

$$\boxed{\sigma(\theta) \equiv \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left|\frac{1}{2\mathrm{i}k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)(\mathbf{S}_L - 1)\right|^2}$$

# QUIZ: What is the value of $S_L$ in the limit of very large L

a) 0
b) 1
c) Infinity
d) Non of the above




$$\mathrm{semi-classical}: \; S(b), \quad \ell = kb$$

# Phase shifts

- Each partial wave S-matrix can be equivalently described with a phase shift

$$\mathbf{S}_L = \mathrm{e}^{2\mathrm{i}\delta_L} \qquad \delta_L(E) = \frac{1}{2\mathrm{i}} \ln \mathbf{S}_L + n(E)\pi$$

added to make the phase shift continuous

- scattering amplitude in terms of phase shifts

$$f(\theta) = \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) P_L(\cos\theta) \mathrm{e}^{\mathrm{i}\delta_L} \sin\delta_L$$

- asymptotic form in terms of phase shift

$$\chi_L^{\mathrm{ext}}(R) \to \mathrm{e}^{\mathrm{i}\delta_L}[\cos\delta_L \sin(kR - L\pi/2) + \sin\delta_L \cos(kR - L\pi/2)]$$
$$= \mathrm{e}^{\mathrm{i}\delta_L} \sin(kR + \delta_L - L\pi/2).$$

# T-matrix

- the partial wave T-matrix is defined as the amplitude of the outgoing wave

$$\chi_L^{\text{ext}}(R) = F_L(0,kR) + \mathbf{T}_L H_L^+(0,kR) \qquad \mathbf{S}_L = 1 + 2\mathrm{i}\mathbf{T}_L$$

- simple relation with the scattering amplitude

$$f(\theta) = \frac{1}{k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)\mathbf{T}_L$$

# QUIZ: What is the value of $T_L$ in the presence of no interactions

a) 0
b) 1
c) Infinity
d) Non of the above

$$\chi_L^{\mathrm{ext}}(R) = F_L(0,kR) + \mathbf{T}_L H_L^+(0,kR)$$

plane wave Scattered wave

# Integrated cross sections

- use properties of legendre polynomials

$$\begin{aligned}
\sigma_{\mathrm{el}} &= \int_0^{2\pi} d\phi \int_0^{\pi} \mathrm{d}\theta \sin\theta\, \sigma(\theta) \\
&= 2\pi \int_0^{\pi} \mathrm{d}\theta \sin\theta |f(\theta)|^2 \\
&= \frac{\pi}{k^2} \sum_{L=0}^{\infty} (2L+1)|1 - \mathbf{S}_L|^2 \\
&= \frac{4\pi}{k^2} \sum_{L=0}^{\infty} (2L+1) \sin^2 \delta_L,
\end{aligned}$$

- Optical theorem:
total elastic cross section related to zero-angle scattering amplitude

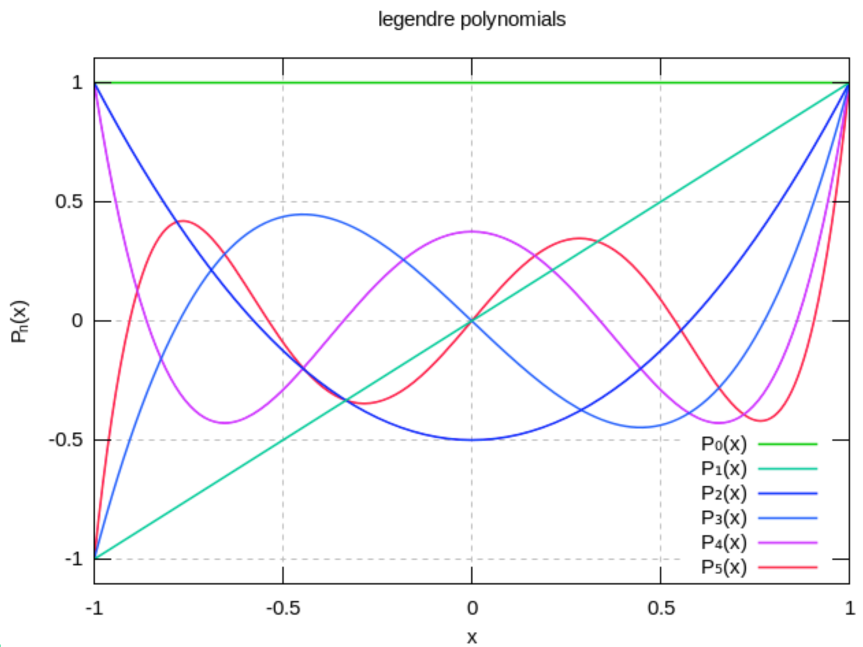
$$f(0) = \frac{1}{2\mathrm{i}k} \sum_{L=0}^{\infty} (2L+1)(\mathrm{e}^{2\mathrm{i}\delta_L} - 1),$$

$$\begin{aligned}
\mathrm{Im} f(0) &= \frac{1}{k} \sum_{L=0}^{\infty} (2L+1) \sin^2 \delta_L \\
&= \frac{k}{4\pi} \sigma_{\mathrm{el}}.
\end{aligned}$$

# T-matrix

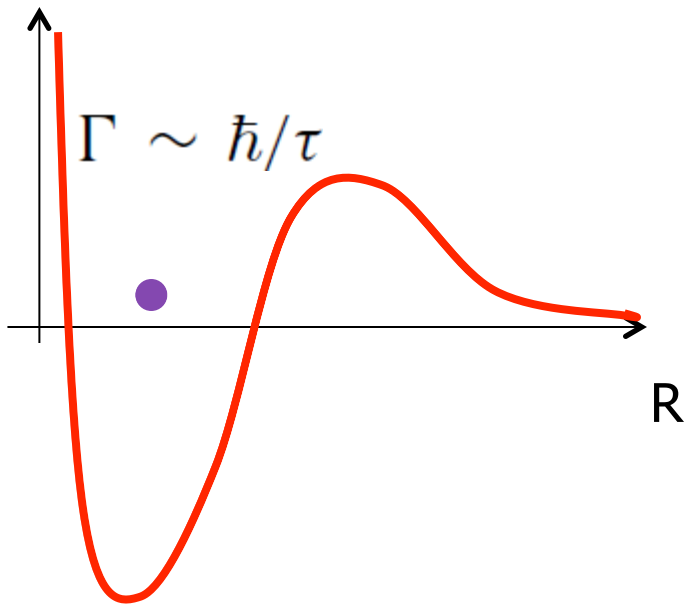
- Contribution of higher partial waves increases the higher the initial momentum

$$f(\theta) = \frac{1}{k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)\mathbf{T}_L$$



# Resonances and phase shifts

- particles trapped inside a barrier




- Resonance characterized by J, E, $\Gamma>0$
- will show rapid rise of phase shift

$$\Delta t \sim \hbar \mathrm{d}\delta(E)/\mathrm{d}E$$

# Resonances and S-matrix

- S-matrix form around the resonance

  (assuming a background in addition to the resonance part)

$$\mathbf{S}(E) = \mathrm{e}^{2\mathrm{i}\delta_{\mathrm{bg}}(E)} \frac{E - E_{\mathrm{r}} - \mathrm{i}\Gamma/2}{E - E_{\mathrm{r}} + \mathrm{i}\Gamma/2}$$

- if analytic continuation to complex energies

  S-matrix **pole** at $\mathbf{E_p = E_r - i\,\Gamma/2}$

$$\delta(E) = \delta_{\mathrm{bg}}(E) + \delta_{\mathrm{res}}(E)$$

$$\delta_{\mathrm{res}}(E) = \arctan\left(\frac{\Gamma/2}{E_{\mathrm{r}} - E}\right) + n(E)\pi$$

- in a pure case, with no background at the resonance energy $\boldsymbol{\delta=\pi/2}$

# Resonances and cross sections

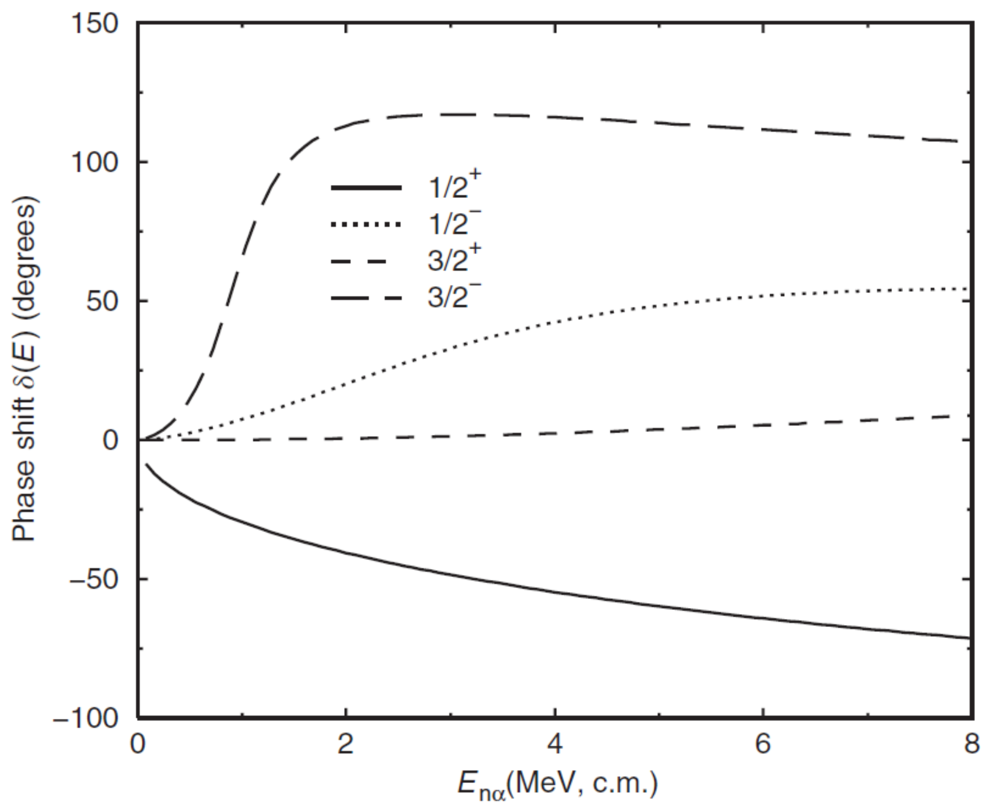



Fig. 3.2. Examples of resonant phase shifts for the $J^\pi = 3/2^-$ channel in low-energy n–$\alpha$ scattering, with a pole at $E = 0.96 - \mathrm{i}0.92/2$ MeV. There is only a hint of a resonance in the phase shifts for the $J^\pi = 1/2^-$ channel, but it does have a wide resonant pole at $1.9 - \mathrm{i}6.1/2$ MeV.

# Complex energy plane

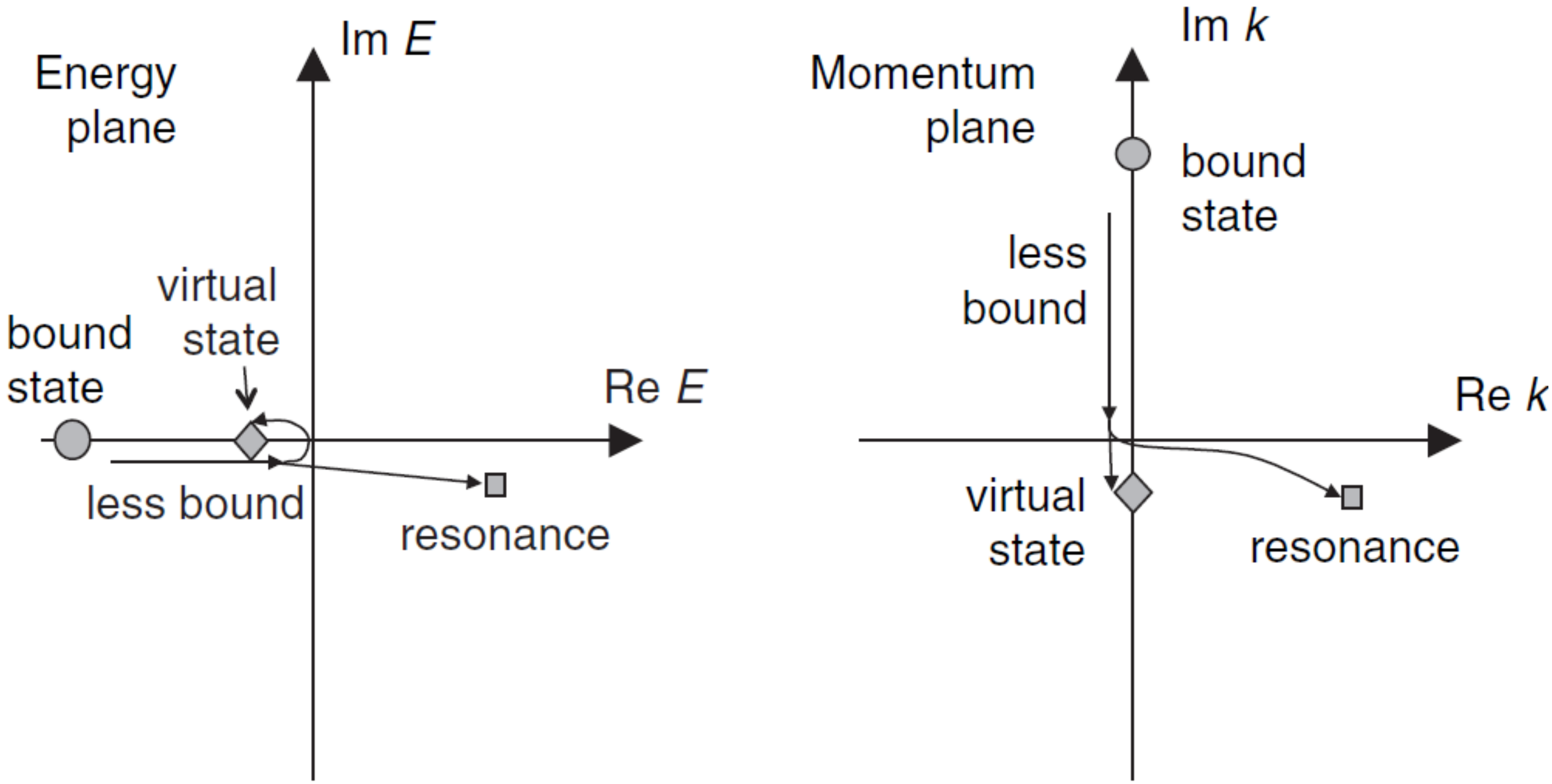


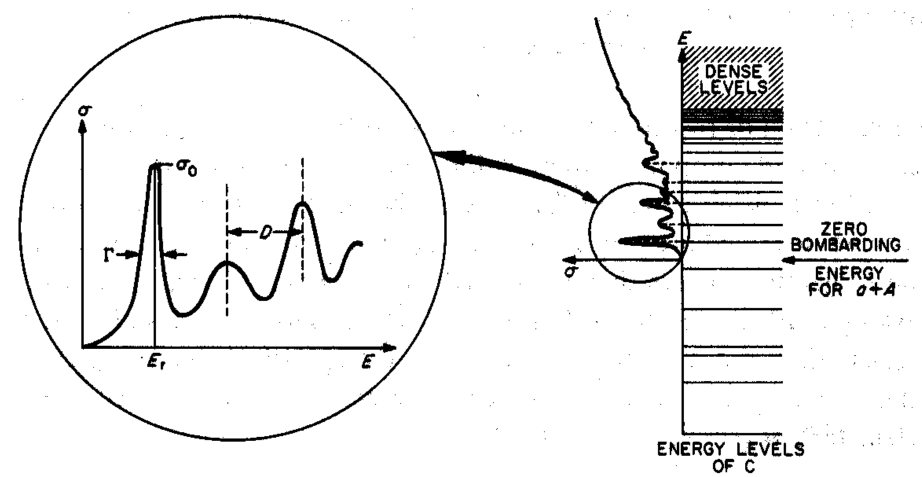

Fig. 3.4. The correspondences between the energy (left) and momentum (right) complex planes. The arrows show the trajectory of a bound state caused by a progressively weaker potential: it becomes a resonance for $L > 0$ or when there is a Coulomb barrier, otherwise it becomes a virtual state. Because $E \propto k^2$, bound states on the positive imaginary $k$ axis and virtual states on the negative imaginary axis both map onto the negative energy axis.

# Resonances and cross sections

- Breit-Wigner form

$$\sigma_{\rm el}^{\rm res}(E) \simeq \frac{4\pi}{k^2}(2L+1)\sin^2\delta_{\rm res}(E)$$

$$= \frac{4\pi}{k^2}(2L+1)\frac{\Gamma^2/4}{(E-E_{\rm r})^2+\Gamma^2/4}$$



# resonance signals

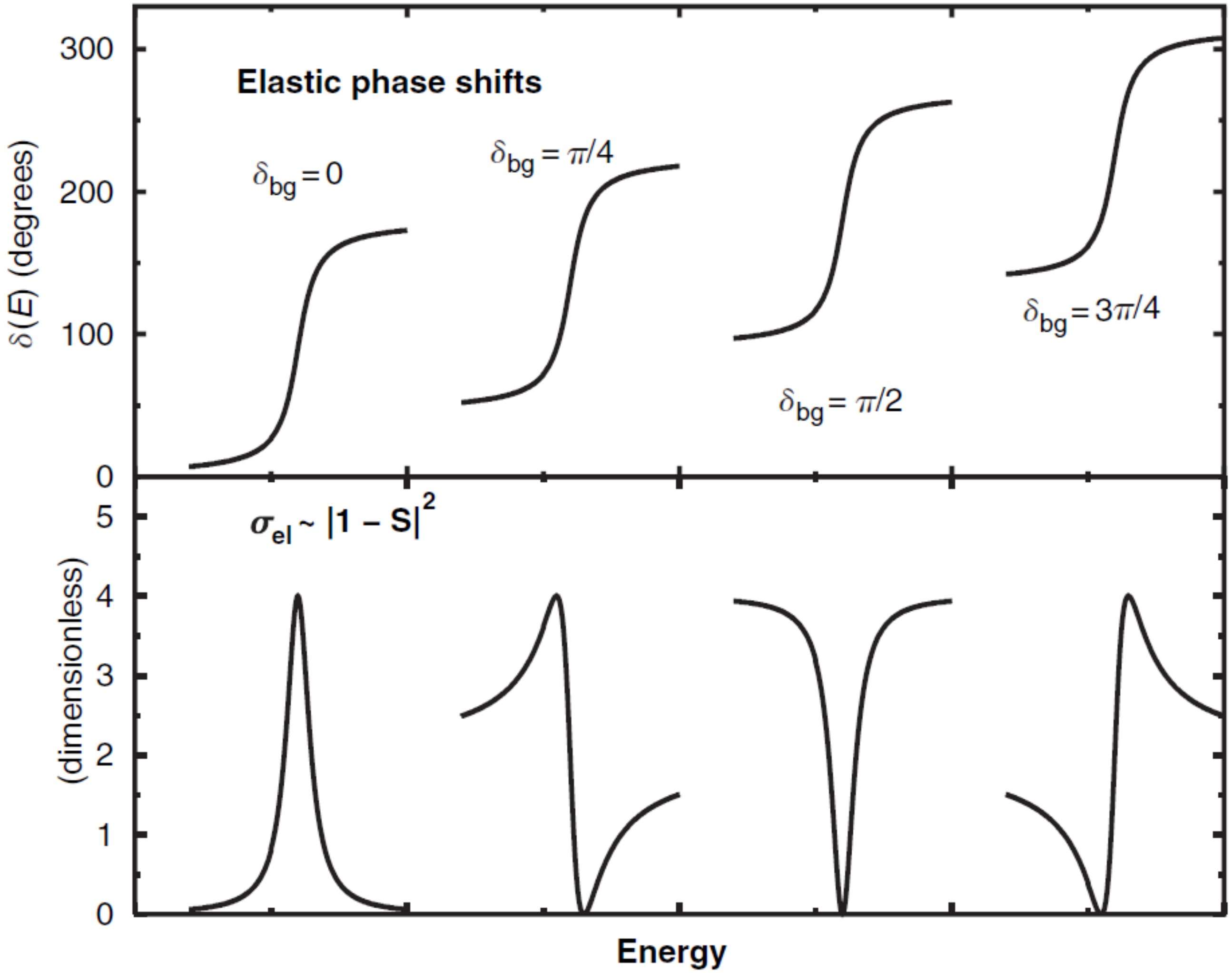



Fig. 3.3. Possible Breit-Wigner resonances. The upper panel shows resonant phase shifts with several background phase shifts $\delta_{bg} = 0, \pi/4, \pi/2$ and $3\pi/4$ in the same partial wave. The lower panel gives the corresponding contributions to the total elastic scattering cross section from that partial wave.

# Scattering theory: single channel

## **Including Coulomb**

# Coulomb scattering – partial wave

- pure Coulomb Schrodinger eq can be solved exactly: $V_c(R) = Z_1 Z_2 e^2/R$

$$\psi_c(\mathbf{k},\mathbf{R}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{R}}\ \mathrm{e}^{-\pi\eta/2}\ \Gamma(1+\mathrm{i}\eta)\ {}_1F_1(-\mathrm{i}\eta;1;\mathrm{i}(kR-\mathbf{k}\cdot\mathbf{R}))$$

- asymptotic form of the scattering wavefunction

$$\psi_c(k\hat{\mathbf{z}},\mathbf{R}) \stackrel{R-Z\to\infty}{\longrightarrow} \mathrm{e}^{\mathrm{i}[kz+\eta\ln k(R-z)]} + f_c(\theta)\frac{\mathrm{e}^{\mathrm{i}[kR-\eta\ln 2kR]}}{R}$$

- without partial wave expansion one can derive the scattering amplitude

$$\boxed{f_c(\theta) = -\frac{\eta}{2k\sin^2(\theta/2)}\exp\left[-\mathrm{i}\eta\ln(\sin^2(\theta/2)) + 2\mathrm{i}\sigma_0(\eta)\right]}$$

Point-Coulomb cross section

$$\boxed{\sigma_{\mathrm{Ruth}}(\theta) = |f_c(\theta)|^2 = \frac{\eta^2}{4k^2\sin^4(\theta/2)}}$$

$\sigma_L(\eta) = \arg\Gamma(1+L+\mathrm{i}\eta)$

# Coulomb+nuclear

- from generalized asymptotic extract the nuclear S-matrix

$$\chi_L^{\text{ext}}(R) = \frac{\mathrm{i}}{2}[H_L^-(\eta, kR) - \mathbf{S}_L^n H_L^+(\eta, kR)]$$

nuclear under the influence of Coulomb

- can be written in terms of the nuclear phase shift

$$\mathbf{S}_L^n = \mathrm{e}^{2\mathrm{i}\delta_L^n}$$

$$\chi_L^{\text{ext}}(R) = \mathrm{e}^{\mathrm{i}\delta_L^n}\left[\cos\delta_L^n\, F_L(\eta, kR) + \sin\delta_L^n\, G_L(\eta, kR)\right]$$

- combined phase shift: Coulomb and "nuclear under the influence of Coulomb"

$$\boxed{\delta_L = \sigma_L(\eta) + \delta_L^n}$$

$$\sigma_L(\eta) = \arg\Gamma(1 + L + \mathrm{i}\eta)$$

# Coulomb+nuclear

Coulomb + "nuclear under Coulomb" phase shifts

$$\boxed{\delta_L = \sigma_L(\eta) + \delta_L^n}$$

$$\mathrm{e}^{2\mathrm{i}\delta_L} - 1 = (\mathrm{e}^{2\mathrm{i}\sigma_L(\eta)} - 1) + \boxed{\mathrm{e}^{2\mathrm{i}\sigma_L(\eta)}(\mathrm{e}^{2\mathrm{i}\delta_L^n} - 1)}$$

$$f_{nc}(\theta) = f_c(\theta) + f_n(\theta)$$

$$\boxed{f_n(\theta) = \frac{1}{2\mathrm{i}k}\sum_{L=0}^{\infty}(2L+1)P_L(\cos\theta)\mathrm{e}^{2\mathrm{i}\sigma_L(\eta)}(\mathbf{S}_L^n - 1)}$$

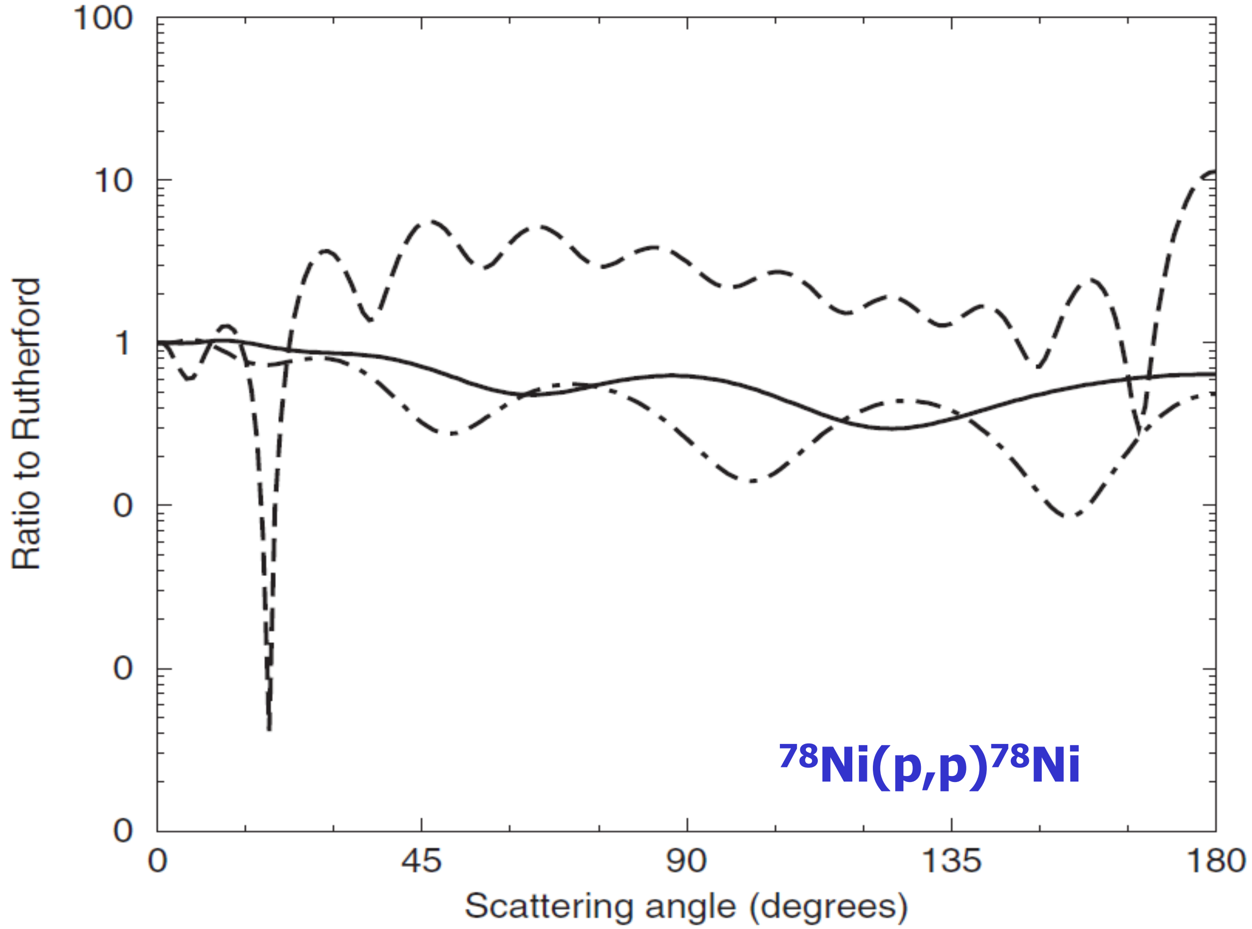
# Coulomb+nuclear scattering

$$\boxed{\sigma_{nc}(\theta) = |f_c(\theta) + f_n(\theta)|^2 \equiv |f_{nc}(\theta)|^2}$$

$$\sigma/\sigma_{\mathrm{Ruth}} \equiv \sigma_{nc}(\theta)/\sigma_{\mathrm{Ruth}}(\theta)$$

Don't add nuclear only and Coulomb only cross sections!

QUIZ: Which distribution corresponds to highest beam energy?

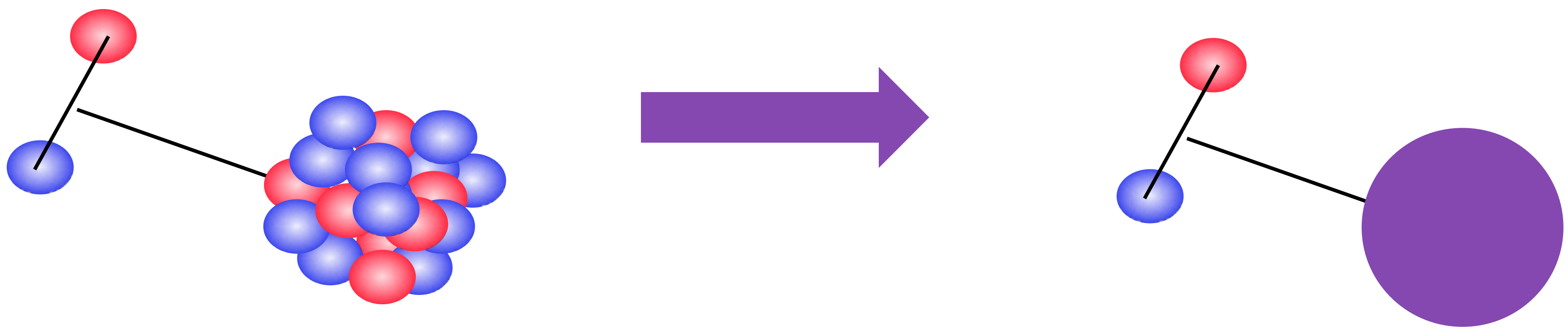



a) Solid
b) Dashed
c) Dot-dashed

## Scattering theory: single channel

## **The optical potential**

# reducing the many body to a few body problem



- ❑ isolating the important degrees of freedom in a reaction
- ❑ keeping track of all relevant channels
- ❑ connecting back to the many-body problem

- ❑ effective nucleon-nucleus interactions – optical potential
  (energy dependence/non-local)
- ❑ many body input

# Optical potential

- Where does the optical potential come from?

Consider the original many-body problem nucleons-nucleus N+A

$$H(\boldsymbol{r}_0;\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)\Psi(\boldsymbol{r}_0;\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A) = E\Psi(\boldsymbol{r}_0;\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)$$

Split the Hamiltonian into:

- kinetic energy of the projectile
- the interaction of the projectile with all nucleons of the target
- internal Hamiltonian of the target

$$H(\boldsymbol{r}_0;\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A) = T_0 + \overset{\mathbf{V}}{\boxed{\sum_{i=1}^{A} V(\boldsymbol{r}_{0i})}} + H_A(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)$$

The solutions for the target Hamiltonian form a complete set:

$$H_A(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)\Phi_i(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A) = \epsilon_i\Phi_i(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)$$

The general solution for N+A can be written in terms of the complete set above:

$$\Psi(\boldsymbol{r}_0;\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A) = \sum_{ij}\chi_i(\boldsymbol{r}_0)\Phi_j(\boldsymbol{r}_1,\boldsymbol{r}_2,\ldots,\boldsymbol{r}_A)$$



*Wong, Introduction to Nuclear Physics, Wiley*

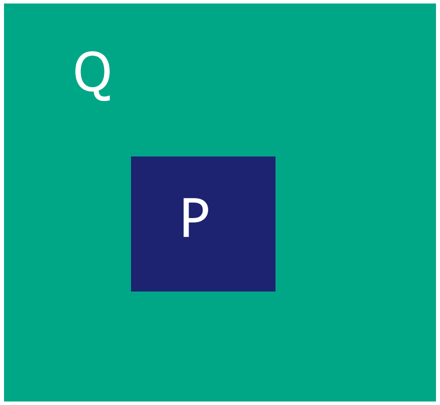
# Optical potential

○ Feshbach projection

Since at this point we still assume in our reaction model that the target stays in the ground state, we need to project the problem into the target ground state.

P is the projection operator: $P = |\Phi_0\rangle\langle\Phi_0|$

It picks up the elastic component: $P\Psi = \chi_0\Phi_0$




Properties of projection operators

$$P^2\Psi = P\Psi$$

$$Q^2\Psi = Q\Psi$$

$$Q = 1 - P$$

$$PQ\Psi = QP\Psi = 0$$

Now apply it to the full equation: $(E - H)(P + Q)\Psi = 0$



*Wong, Introduction to Nuclear Physics, Wiley*

# Optical potential

- The scattering equation can be rewritten: $(E - T_0 - \mathcal{V}(\boldsymbol{r}_0))\chi_0 = 0$

  with the effective potential: $$\mathcal{V}(\boldsymbol{r}_0) = \langle\Phi_0|V|\Phi_0\rangle + \langle\Phi_0|VQ\frac{1}{E - QHQ}QV|\Phi_0\rangle$$

- This potential is generally non-local which gives rise to some complications:

$$(E - T_0)\chi_0(\boldsymbol{r}_0) = \mathcal{V}(\boldsymbol{r}_0)\chi_0(\boldsymbol{r}_0) + \int f(\boldsymbol{r}_0, \boldsymbol{r}_0')\chi_0(\boldsymbol{r}_0')d\boldsymbol{r}_0'$$

Often this is approximated to a local version.
The optical model replaces this microscopic potential by a model potential obtained phenomenologically: $(E - T_0 - U_{\text{opt}})\chi_0 = 0$

Scattering into Q-space may never return to elastic – loss of flux

Optical potential needs to have an imaginary term!



*Wong, Introduction to Nuclear Physics, Wiley*

# Optical potentials

- to account for other processes – introduce imaginary part in interaction

$$V_c(R) + V(R) + i\,W(R) + V_{so}(R)$$

- loss of flux - absorption (W<0)

- Nucleon potentials as described with Woods-Saxon shape (to mimic the density distribution in nuclei)

$$V(R) = -\frac{V_r}{1+\exp\left(\frac{R-R_r}{a_r}\right)}$$

$$W(R) = -\frac{W_i}{1+\exp\left(\frac{R-R_i}{a_i}\right)}$$

- Sometimes imaginary also defined at $d/dR(V_{ws}(r))$ - surface

$$R_r = r_r A^{1/3}$$

For nucleon-nucleus interactions V=40-50 MeV, r=1.2 fm and a=0.6-0.65 fm

# reaction and absorptive cross section

$$\sigma_R = \frac{\pi}{k^2} \sum_L (2L+1)(1 - |\mathbf{S}_L|^2)$$

$$\sigma_A = \frac{2}{\hbar v} \frac{4\pi}{k^2} \sum_L (2L + 1) \int_0^\infty [-W(R)] \, |\chi_L(R)|^2 \, \mathrm{d}R$$

For simple spherical potentials, reaction and absorptive cross sections are the same

BACKUP

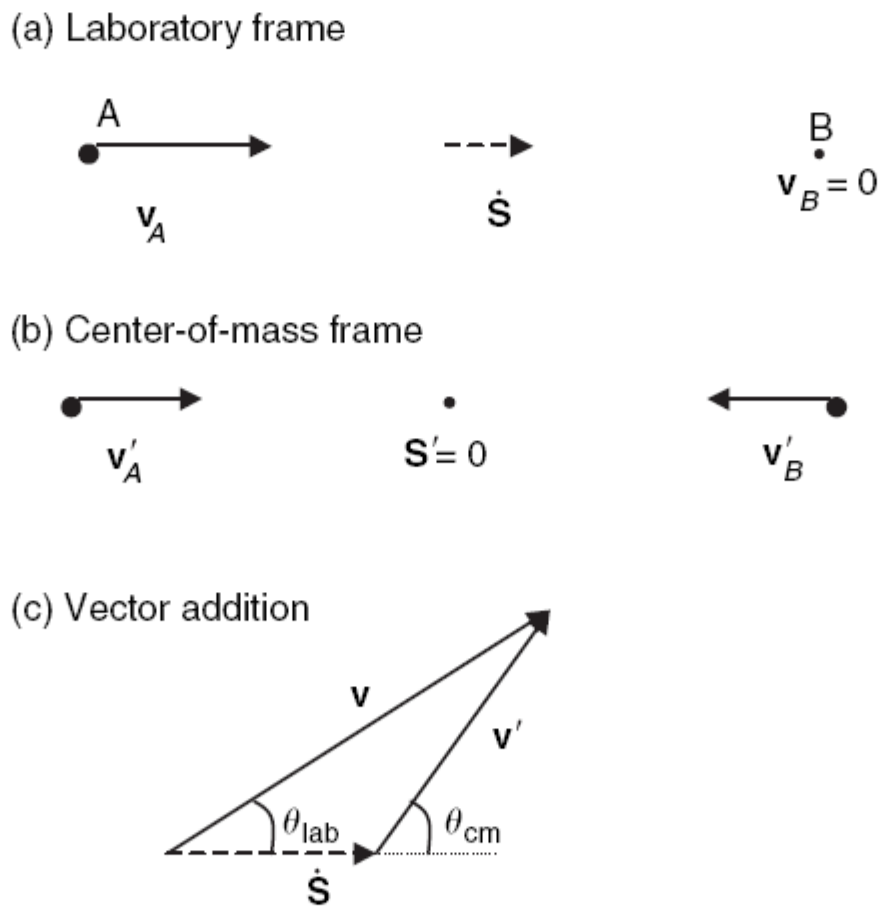
# kinematic of reactions

$$E_{\text{tot}} = \frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2$$

$$E_{\text{tot}} = \frac{1}{2}m_{AB}\dot{S}^2 + \frac{1}{2}\mu\dot{R}^2$$

energy in the relative motion

$$E = m_B/m_{AB}\, E_A = \frac{1}{2}\mu v_A^2.$$



# kinematics of the reaction

apply laws of conservation

conservation of mass

conservation of energy

consevation of momentum

$$m_A + m_B = m_C + m_D,$$

$$Q + E_A + E_B = E_C + E_D,$$

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}_C + \mathbf{p}_D,$$

# Properties of Coulomb waves with η=0

The $\eta = 0$ functions are more directly known in terms of Bessel functions:

$$F_L(0,\rho) = \rho\, j_L(\rho) = (\pi\rho/2)^{1/2} J_{L+1/2}(\rho)$$

$$G_L(0,\rho) = -\rho\, y_L(\rho) = -(\pi\rho/2)^{1/2} Y_{L+1/2}(\rho),$$

Their behaviour near the origin, for $\rho \ll L$, is

$$F_L(0,\rho) \sim \frac{1}{(2L+1)(2L-1)\cdots 3.1}\rho^{L+1}$$

$$G_L(0,\rho) \sim (2L-1)\cdots 3.1\ \rho^{-L},$$

and their asymptotic behaviour when $\rho \gg L$ is

$$F_L(0,\rho) \sim \sin(\rho - L\pi/2)$$

$$G_L(0,\rho) \sim \cos(\rho - L\pi/2)$$

$$H_L^{\pm}(0,\rho) \sim \mathrm{e}^{\pm\mathrm{i}(\rho - L\pi/2)} = \mathrm{i}^{\mp L}\mathrm{e}^{\pm\mathrm{i}\rho}.$$

So $H_L^+$ describes an outgoing wave $\mathrm{e}^{\mathrm{i}\rho}$, and $H_L^-$ an incoming wave $\mathrm{e}^{-\mathrm{i}\rho}$.

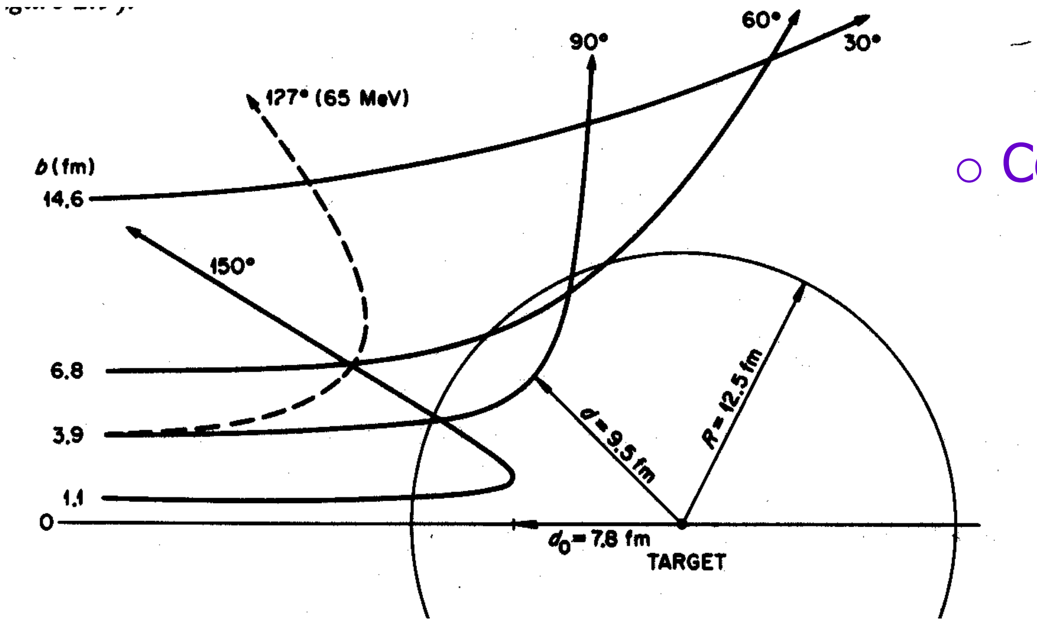
# Relations between T, S, δ

| Using: | $\delta$ | $\mathbf{T}$ | $\mathbf{S}$ |
|---|---|---|---|
| $\chi(R) =$ | $e^{i\delta}[F\cos\delta + G\sin\delta]$ | $F + \mathbf{T}H^{+}$ | $\frac{i}{2}[H^{-} - \mathbf{S}H^{+}]$ |
| $\delta =$ | $\delta$ | $\arctan\dfrac{\mathbf{T}}{1+i\mathbf{T}}$ | $\frac{1}{2i}\ln\mathbf{S}$ |
| $\mathbf{T} =$ | $e^{i\delta}\sin\delta$ | $\mathbf{T}$ | $\frac{i}{2}(1-\mathbf{S})$ |
| $\mathbf{S} =$ | $e^{2i\delta}$ | $1 + 2i\mathbf{T}$ | $\mathbf{S}$ |
| $V = 0$ | $\delta = 0$ | $\mathbf{T} = 0$ | $\mathbf{S} = 1$ |
| $V$ real | $\delta$ real | $\lvert 1 + 2i\mathbf{T}\rvert = 1$ | $\lvert\mathbf{S}\rvert = 1$ |

# Virtual states

- neutral L=0 particles: no barrier
- S-matrix pole is on negative imaginary k-axis (not a bound state!)
- scattering length $k_p = \mathrm{i}/a_0$ $\quad k_p = \pm\sqrt{2\mu E_p/\hbar^2}$
- S-matrix in terms of scattering length $$\mathbf{S}(k) = -\frac{k + \mathrm{i}/a_0}{k - \mathrm{i}/a_0}$$
- phase shift in terms of scattering length $$k \cot\delta(k) = -1/a_0$$

# Classical Coulomb scattering

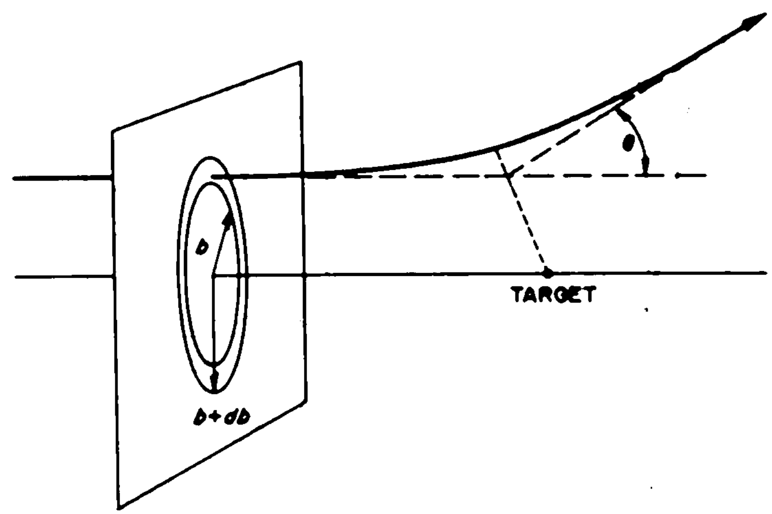



- Coulomb trajectories are hyperbolas

$$\tan\frac{\theta}{2} = \frac{\eta}{bk}$$

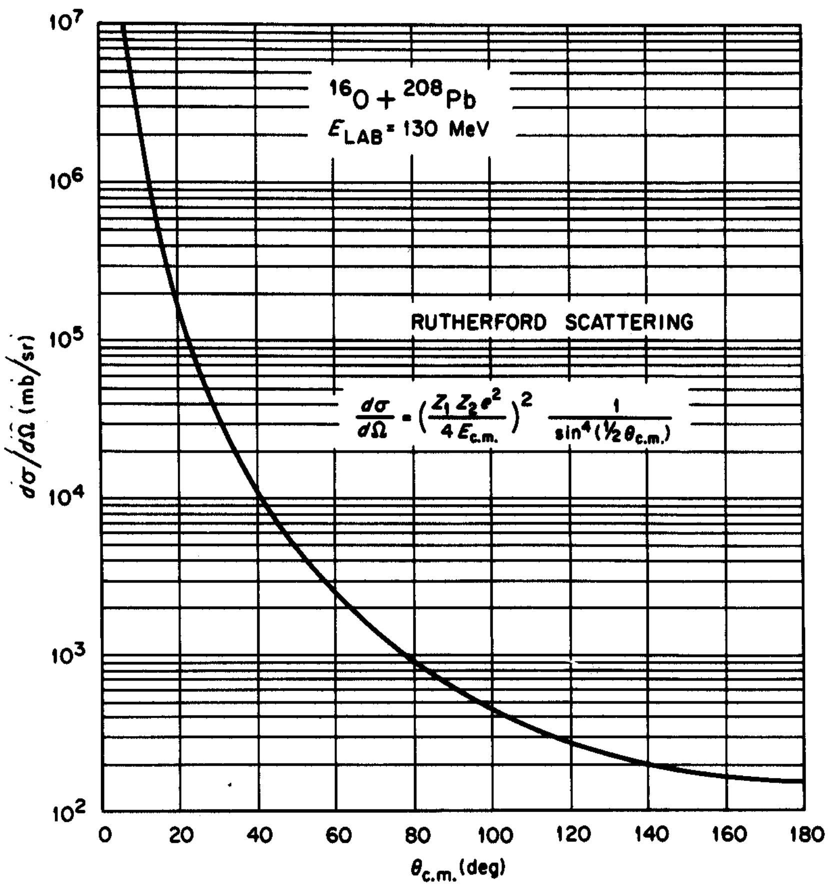
- the cross section for a pure Coulomb interaction is

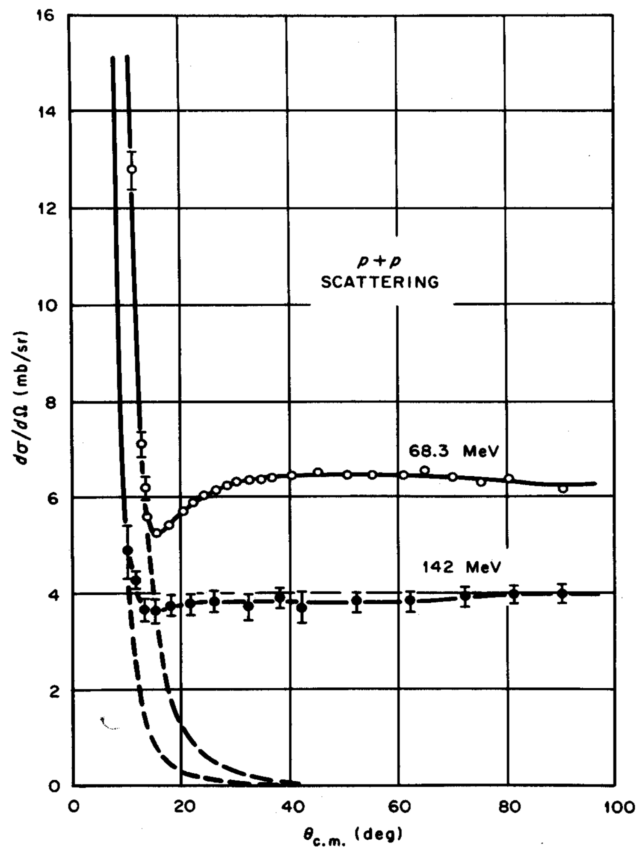
$$\sigma(\theta) \equiv \frac{b(\theta)}{\sin\theta}\frac{db}{d\theta} = \frac{\eta^2}{4k^2\sin^4(\theta/2)}$$



# Coulomb scattering

- examples



# Coulomb functions

○ Coulomb wave equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}\rho^2} - \frac{L(L+1)}{\rho^2} - \frac{2\eta}{\rho} + 1\right] X_L(\eta,\rho) = 0$$

$$F_L(\eta,\rho) = C_L(\eta)\rho^{L+1}\mathrm{e}^{\mp\mathrm{i}\rho}\ {}_1F_1(L+1\mp\mathrm{i}\eta; 2L+2; \pm 2\mathrm{i}\rho)$$

$$C_L(\eta) = \frac{2^L\mathrm{e}^{-\pi\eta/2}|\Gamma(1+L+\mathrm{i}\eta)|}{(2L+1)!}$$

$${}_1F_1(a;b;z) = 1 + \frac{a}{b}\frac{z}{1!} + \frac{a(a+1)}{b(b+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{b(b+1)(b+2)}\frac{z^3}{3!} + \cdots .$$

$$\begin{aligned} H_L^{\pm}(\eta,\rho) &= G_L(\eta,\rho) \pm \mathrm{i}F_L(\eta,\rho) \\ &= \mathrm{e}^{\pm\mathrm{i}\Theta}(\mp 2\mathrm{i}\rho)^{1+L\pm\mathrm{i}\eta}U(1+L\pm\mathrm{i}\eta, 2L+2, \mp 2\mathrm{i}\rho) \end{aligned}$$

$$\Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta\ln(2\rho)$$

$$\sigma_L(\eta) = \arg\Gamma(1+L+\mathrm{i}\eta)$$

# Coulomb functions

## ○Behaviour near the origin

$$F_L(\eta,\rho) \sim C_L(\eta)\rho^{L+1}, \quad G_L(\eta,\rho) \sim \left[(2L+1)C_L(\eta)\,\rho^L\right]^{-1}$$

$$C_0(\eta) = \sqrt{\frac{2\pi\eta}{e^{2\pi\eta}-1}} \quad \text{and} \quad C_L(\eta) = \frac{\sqrt{L^2+\eta^2}}{L(2L+1)} C_{L-1}(\eta)$$

A transition from small-$\rho$ power law behavior to large-$\rho$ oscillatory behavior occurs outside the classical turning point. This point is where $1 = 2\eta/\rho + L(L+1)/\rho^2$, namely

$$\rho_{\mathrm{tp}} = \eta \pm \sqrt{\eta^2 + L(L+1)}. \tag{3.1.67}$$

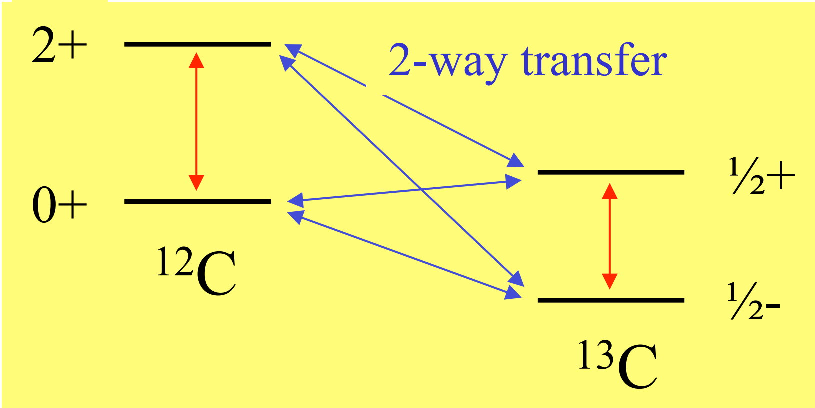
## ○Behaviour at large distances

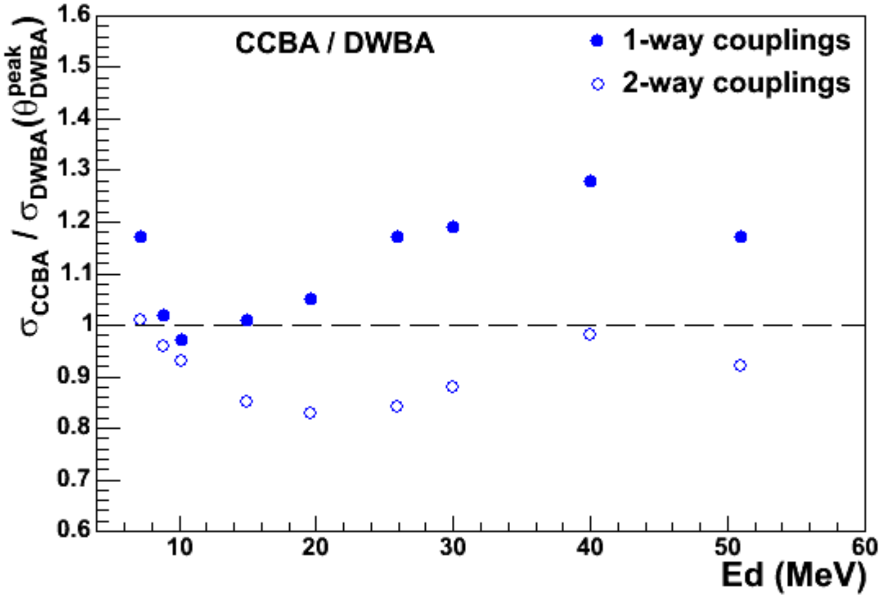
$$F_L(\eta,\rho) \sim \sin\Theta, \quad G_L(\eta,\rho) \sim \cos\Theta, \quad \text{and} \quad H_L^{\pm}(\eta,\rho) \sim e^{\pm i\Theta}$$

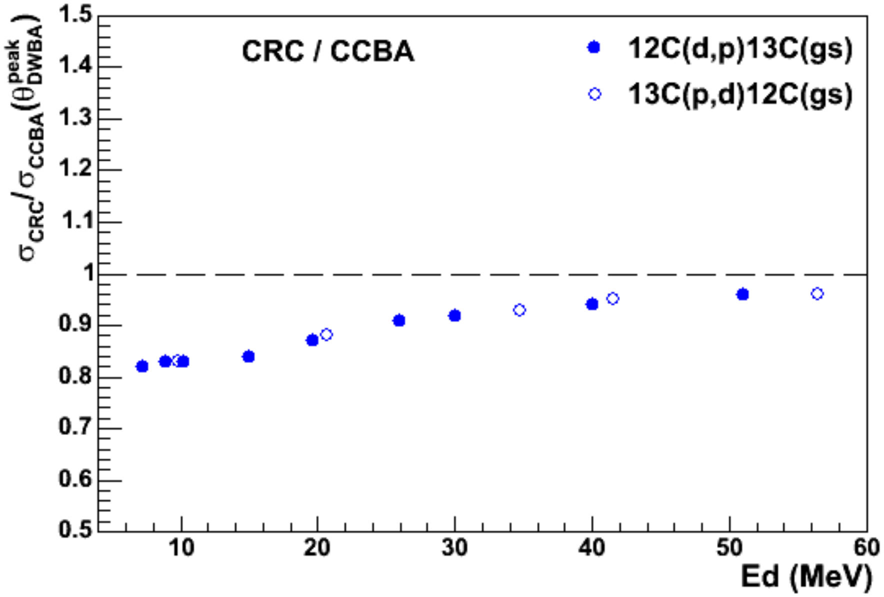
$$\Theta \equiv \rho - L\pi/2 + \sigma_L(\eta) - \eta\ln(2\rho)$$

# Reaction mechanism: target excitations

$^{12}C(d,p)^{13}C$: couplings




Delaunay et al, PRC 72 (2005) 014610